The Dynamics of Working Hours and Wages Under Implicit Contracts*

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Abstract

In this paper, we explore the out-of-equilibrium dynamics of working hours and wages in a model economy where workers and firms have agreed on an implicit contract that smooths long-run consumption. Specifically, we develop a deterministic and a stochastic framework in which a firm inter-temporally sets its level of labour utilization by considering that workers' earnings tend to adjust in the direction of a fixed level that seeks to stabilize their consumption. Without any uncertainty in labour effectiveness, this theoretical setting may have one, two or none stationary solution. However, the dynamics of the deterministic economy can be assessed only in the two-solution case and it reveals that wages move counter-cyclically towards the allocation preferred by the firm. Adding uncertainty in labour effectiveness does not overturn the counter-cyclical pattern of wages but is helpful in explaining the wage stickiness observed at the macro level.

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1 Introduction

The theory of implicit contracts starts from the premise that the labour market is far from being a spot market but, on the contrary, workers and firms usually display the tendency to be involved in long-lasting relationships (cf. Okun, 1981). Consequently, if there is some kind of uncertainty about actual production outcomes and entrepreneurs are more risk-prone than workers, then it may happen that the two parties will consensually rely on informal agreements

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on wage payments and labour provisions that optimally share the burden of realized labour income fluctuations (cf. Baily, 1974; Gordon, 1974; Azariadis, 1975).

The theoretical literature on implicit contracts collects a number of contributions in which labour market outcomes are determined in a time-less perspective (e.g. Geanakoplos and Ito, 1982; Azariadis and Stiglitz, 1983; Bull, 1983, 1987; Chiari, 1983; Baker et al. 1997). In more recent years, however, after the seminal work by Haltiwanger and Maccini (1985) in which implicit contracts may lead firms to rely on temporary layoffs and recalls, some authors put some effort to exploring the dynamic consequences on hours, (un)employment and wages triggered by the existence of optimal risk-sharing in labour contracts (e.g. Robinson, 1999; Calmès, 2007; Gürtler, 2006; Wang, 2015).

In this paper, we aim at contributing to the literature on dynamic implicit-contract models by deriving the out-of-equilibrium dynamics of working hours and wages in a theoretical setting where workers and firms have agreed upon a self-enforcing implicit contract that seeks to smooth real consumption in a long-run perspective (cf. Romer, 2019, Chapter 11). Specifically, we develop a theoretical framework in which a representative firm inter-temporally sets its optimal level of labour utilization by taking into account that workers' earnings tends to adjust in the direction of a fixed level quoted in the contract that is assumed to coincide with long-run consumption (cf. Abowd and Card, 1987).

Our theoretical exploration is split up in two parts. First, we explore the out-of-equilibrium dynamics of hours and wages in a model economy with no uncertainty in labour productivity. Thereafter, we consider the optimal adjustments of hours and wages by assuming that the effectiveness of labour is hit by random shocks that systematically alter the profitability of the firm. The former case allows us to discuss the conditions under which the suggested contractual agreement between the firm and its workers conveys meaningful solutions. The latter provides the basis to assess the cyclical properties of the model economy under investigation.

Overall, our analysis provides a number of interesting findings. On the one hand, depending on selected parameter values, the deterministic model may have one, two or none stationary solution. Interestingly, whenever there are two steady-state allocations for hours and wages the resting points of the economy without uncertainty can be ordered according to the preferences of each party. Moreover, in the two-solution case, the local dynamics of the model reveals that wages display the tendency to move in the opposite direction with respect to working hours by converging towards the allocation preferred by the firm and loathed by its workers. This result is consistent with the empirical tests of the implicit contract theory carried out in the US at the micro level by Beaudry and DiNardo (1995); indeed, in their pioneering study - controlling for productivity - higher wages appear associated with lower hour provision. In addition, when the initial contract wage overshoots (undershoots) its long-equilibrium value, workers' consumption remains above (below) the contract fixed level during the whole adjustment process.

In the other hand, simulations of the stochastic model run by targeting the observed volatility of output show that productivity disturbances does not overturn the counter-cyclical pattern of wages. Moreover, we show the insurance scheme underlying the dynamic implicit contract displays the tendency to underestimate the volatility of labour earnings but it has the potential to explain the degree of real-wage stickiness usually observed in macro data (cf. Ravn and Simonelli, 2007).

This paper is arranged as follows. Section 2 describes the theoretical setting. Section 3 develops the deterministic model. Section 4 analyzes the model with uncertainty in the effectiveness of labour. Finally, section 5 concludes.

2 Theoretical setting

We consider a model economy in which time is continuous and a representative risk-neutral firm deals with a group of risk-averse identical workers that cannot purchase insurance on the level of their long-run labour income. Within this environment, given the different attitude towards risk, we make the hypothesis that the firm and the workers have implicitly agreed upon a self-enforcing wage contract that seeks to stabilize the level of long-run labour earnings. Assuming the absence of non-labour income and saving on the side of workers, this means that the informal - but binding - agreement between the workers and the firm will tend to stabilize real consumption in a long-run perspective (cf. Abowd and Card, 1987).

On the productive side - similarly to Guerrazzi (2011, 2016) - we assume that the representative firm is endowed with a quadratic production function so that instantaneous output $Y(t)$ is equal to

$$Y(t) = A(t) L(t) - \frac{1}{2} (L(t))^2 \tag{1}$$

where $A(t) > 0$ is technology variable taken as given by the firm and its workers whereas $L(t)$ is the labour provision of the workers attached to the firm measured in hours.

The variable $A(t)$ in eq. (1) affects the marginal productivity of employed workers and it conveys the actual realization of the state of the world. Specifically, the higher (lower) the value of $A(t)$, the higher (lower) the marginal productivity of employed workers so that the better (worse) the realized state of the world. In the remainder of the paper, we assume that $A(t)$ moves over time according to an Ornstein-Uhlenbeck process (cf. Cox and Miller, 1967). Formally speaking, this means that

$$\dot{A}(t) = \kappa(\mu_A - A(t)) + \sigma_A \dot{x}(t) \tag{2}$$

where $\mu_A > 0$ is the long-run mean of the process, $\kappa > 0$ is its speed of mean reversion, $\sigma_A > 0$ is its instantaneous standard deviation whereas $\dot{x}(t)$ is a standard Brownian motion with zero drift and unit variance.

As implied by the text-book treatment offered by Romer (2019, Chapter 11), in a timeless contracting model with stochastic productivity, the fixed level of consumption granted to workers in all the states of the world can be conveyed as a non-linear combination of the possible realizations of productivity shocks whose weights are affected by workers preferences, the available productive technology and the probability distribution of the already mentioned productivity shocks.[1] Consequently, under the assumption that the information on these fundamental factors is costlessly available to all the involved parties of the contract and agents are rational, such a critical level of consumption can be taken as exogenously given without any loss of generality.

Along these lines, in the remainder of this paper we will assume that the long-run consumption granted to employed workers who have agreed upon the dynamic implicit contract is fixed at the constant level $C > 0$. Thereafter, the out-of-equilibrium dynamics of the contract real wage $w(t)$ aimed at equalizing in the long-run the wage bill to C is given by

$$\dot{w}(t) = \theta \left(\frac{C}{L(t)} - w(t) \right) \tag{3}$$

where $\theta > 0$ is a measure of the attrition between the actual and the long-run real wage that stabilizes consumption.

The expression in eq. (3) implies that in each instant the contract wage increases (decreases) whenever it is below (above) the long-run consumption per unit of hours.[2] Such a differential equation is not affected by the evolution of $A(t)$ in order to capture the idea that wage contract is not renegotiated when the state of the world changes. Furthermore, the parameter θ in the RHS of eq. (3) can be taken as a measure of the degree of aversion with respect to situations of under- or overconsumption; indeed, for any given level of L, the higher (lower) the value of θ, the faster w adjust itself in the direction of C.

Following the game-theoretical arguments put forward by Bull (1987), the wage trajectory implied by eq. (3) can be thought as the outcome of a Nash equilibrium of a post-hiring trading game whose self-enforceability is supported by intra-firm reputation. In other words, if workers provide the required amount of hours and the firm deviates from the wage trajectory conveyed by eq. (3), then it will be punished its employees with shirking and it will signal itself as a cheater by preventing the formation of long-lasting relationships in future periods. Consequently, after the agreement has been achieved, no party has incentive to renege (cf. Thomas and Worrall, 1988).

3 The deterministic model

We begin our analysis by considering what happens in model economy without uncertainty so that the state of the world is revealed to the firm and its workers and is assumed to remain constant over time. Specifically, we initially assume that

$$A(t) = A \quad \text{for all } t \tag{4}$$

[1] A formal proof for this statement is sketched in Appendix.

[2] As we show in Appendix, assuming that the real wage increases (decreases) when labour earnings are below (above) the long-run level of consumption quoted in the contract complicates the analytical treatment of the model without any substantial modification in final conclusions achieved in the paper.

Given the values of C, θ, and A, the inter-temporal problem of the representative firm in the model economy described above is to set the optimal labour input in each instant aiming at maximizing its profits by tacking into account that the real wage adjusts according to the differential equation in (2). Formally speaking, taking into account the production function in eq. (1) and the simplifying assumption in (4), the problem of the representative firm is the following:

$$\begin{gathered}\max_{\{L(t)\}_{t=0}^{\infty}} \int_{t=0}^{\infty} \exp(-\rho t) \left(AL(t) - \tfrac{1}{2} (L(t))^2 - w(t) L(t) \right) dt \\ \text{s.to} \\ \dot{w}(t) = \theta \left(\tfrac{C}{L(t)} - w(t) \right) \\ w(0) = w_0 \end{gathered} \tag{5}$$

where $\rho > 0$ is the discount rate of entrepreneurs, whereas $w_0 > 0$ is the initial level of the real wage rate quoted on the contract.

The first-order conditions (FOCs) of the problem in (5) can be written as

$$A - L(t) - w(t) - \theta C \frac{\Lambda(t)}{(L(t))^2} = 0 \tag{6}$$

$$\dot{\Lambda}(t) = (\rho + \theta) \Lambda(t) + L(t) \tag{7}$$

$$\lim_{t \to \infty} \Lambda(t) w(t) = 0 \tag{8}$$

where $\Lambda(t)$ is the costate variable associated to $w(t)$.

Eq. (6) is the FOC with respect to the control variable of the firm, that is, $L(t)$. Moreover, the differential equation in (7) describes the optimal path of $\Lambda(t)$, whereas (8) is the required transversality condition.

After a trivial manipulation, the results in eq.s (6) and (7) allow us to obtaining the following differential equation for the out-of-equilibrium dynamics of working hours:

$$\dot{L}(t) = \frac{(\rho + \theta) L(t) (A - L(t) - w(t)) + \theta (2C - w(t) L(t))}{2(A - L(t) - w(t)) - L(t)} \tag{9}$$

Starting from given initial conditions to be defined, the differential equations in (3) and (9) describe how hours of work and wages move over time once an everlasting state of the world is revealed to the firm and its workers. Consequently, eq.s. (3) and (9) convey the dynamics of wages and hours for a given level of labour productivity.

3.1 Steady-state equilibria

Within the model under investigation, steady-state equilibria are defined as the set of pairs $\{L^*, w^*\}$ such that $\dot{L}(t) = \dot{w}(t) = 0$. Obviously, the elements of that set are given by allocations in which the real wage bill equals the fixed level of consumption fixed by the implicit contract on which the firm and its workers agreed upon.

From a formal point of view, the derivation of $\{L^*, w^*\}$ is straightforward. First, setting $\dot{w}(t) = 0$ in eq. (3) leads to

$$w^* = \frac{C}{L^*} \tag{10}$$

Thereafter, setting $\dot{L}(t) = 0$ in eq. (9) and plugging the result in eq. (10) leads to the following quadratic expression:

$$(L^*)^2 - AL^* + \frac{C\rho}{\rho + \theta} = 0 \tag{11}$$

The parabola in eq. (11) allows to state the following three plain propositions:

Proposition 1: When $A = 2\sqrt{C\rho/(\rho+\theta)}$, there is only one stationary solution given by $L_0^* \equiv A/2$ and $w_0^* \equiv 2C/A$.

Proposition 2: When $A > 2\sqrt{C\rho/(\rho+\theta)}$, there are two distinct stationary solutions given by $L_1^* \equiv 1/2\left(A - \sqrt{A^2 - 4C\rho/(\rho+\theta)}\right)$ and $w_1^* \equiv 2C/\left(A - \sqrt{A^2 - 4C\rho/(\rho+\theta)}\right)$ as well as $L_2^* \equiv 1/2\left(A + \sqrt{A^2 - 4C\rho/(\rho+\theta)}\right)$ and $w_2^* \equiv 2C/\left(A + \sqrt{A^2 - 4C\rho/(\rho+\theta)}\right)$.

Proposition 3: When $A < 2\sqrt{C\rho/(\rho+\theta)}$, there are no (real) stationary solutions.

Proposition 1 provides the parameters' combination under which there is a unique steady-state (L_0^*, w_0^*). In that allocation, equilibrium hours are an increasing function of the parameter that conveys the actual state of the world, whereas the equilibrium wage increases (decreases) with the fixed level of consumption granted by the implicit contract (the realized state of the world) virtually signed by the firm and its employees.[3] This pattern clearly points out the insurance component of the implicit contract under which workers tend to work more (less) for less (more) in good (bad) states.

By contrast, Proposition 2 reveals the condition under which - similarly to what happens in the dynamic model by Diamond (1985) - there are two different steady-states, that is, (L_1^*, w_1^*) and (L_2^*, w_2^*).[4] Assuming separability between leisure and consumption in the utility function of workers, the two stationary solutions pointed out in Proposition 2 can be ordered according to the preferences of the involved economic agents. Specifically, since the implied level of consumption - or the implied labour earnings - is the same in both allocations, (L_1^*, w_1^*), that is, the stationary solution with low equilibrium hours and high equilibrium wage, is the most preferred by workers because it implies more leisure, whereas (L_2^*, w_2^*), that is, the stationary

[3]It is worth noticing that the unique stationary solution falls in the concave part of the production function in eq. (1).

[4]Obviously, for (L_1^*, w_1^*) to be feasible it must hold that $A > \sqrt{A^2 - 4C\rho/(\rho+\theta)}$. In the remainder of the paper, we will assume that when the condition pointed out by Proposition 2 is met such an inequality is always fulfilled.

solution with high equilibrium hours and low equilibrium wage, is most the preferred by the firm because - everything else being equal - it implies higher profits.[5]

Furthermore, Proposition 3 shows the condition under which a steady-state does not exist. For a given value of the state of the world conveyed by A, the impossibility to retrieving a stationary solution for the dynamics of working hours and wages appears alternatively related to an excessive degree of impatience on the side of the firms mirrored in the value taken by ρ, to an excessive fixed level of long-run consumption granted to workers embodied in the actual level of C or to an excessive rate of mean reversion of contract wages conveyed by the value of θ.

3.2 Local dynamics

Given the stationary solution $\{L^*, w^*\}$, the local dynamics of working hours and wages is given by the following 2×2 linear system:

$$\begin{pmatrix} \dot{L}(t) \\ \dot{w}(t) \end{pmatrix} = \begin{bmatrix} j_{1,1} & j_{1,2} \\ -\frac{\theta C}{(L^*)^2} & -\theta \end{bmatrix} \begin{pmatrix} L(t) - L^* \\ w(t) - w^* \end{pmatrix} \tag{12}$$

where $j_{1,1} \equiv \partial \dot{L}(t) / \partial L(t) \Big|_{L(t)=L^*, w(t)=w^*}$ and $j_{1,2} \equiv \partial \dot{L}(t) / \partial w(t) \Big|_{L(t)=L^*, w(t)=w^*}$.

In general terms, the two unspecified elements on the first row of the Jacobian matrix in (12) can be written as

$$j_{1,1} = \frac{\left((\rho+\theta)\Phi(L^*) - \frac{\theta C}{L^*}\right)(2\Gamma(L^*) - L^*) + 3\left((\rho+\theta)\left(AL^* - (L^*)^2 - C\right) + \theta C\right)}{(2\Gamma(L^*) - L^*)^2} \tag{13}$$

$$j_{1,2} = \frac{2\left((\rho+\theta)\left(AL^* - (L^*)^2 - C\right) + \theta C\right) - (\rho + 2\theta) L^* (2\Gamma(L^*) - L^*)}{(2\Gamma(L^*) - L^*)^2} \tag{14}$$

where $\Phi(L^*) \equiv \left(AL^* - 2(L^*)^2 - C\right)/L^*$ and $\Gamma(L^*) \equiv \left(AL^* - (L^*)^2 - C\right)/L^*$.

Under the condition pointed out in Proposition 1, that is, when there is only one stationary solution given by (L_0^*, w_0^*), the Jacobian matrix of the system in (12) merely reduces to

$$\begin{bmatrix} \rho + \theta & \rho \\ -\frac{\theta(\theta+\rho)}{\rho} & -\theta \end{bmatrix} \tag{15}$$

The trace of the matrix in (15) is equal to ρ whereas its determinant is equal to zero. This means that one eigenvalue of system is zero whereas the other is equal to ρ. Consequently, when the parameters of the model deliver a unique stationary solution the out-of-equilibrium dynamics of hours and wages cannot be assessed; indeed, this is a degenerate case in which

[5] To some extent, $A(L_2^* - L_1^*) - 1/2\left((L_2^*)^2 - (L_1^*)^2\right)$ can be taken as a proxy of the equilibrium reward that the firm receives for its insurance service. In a similar manner, if $U(C) - V(L)$ is utility function of workers, their equilibrium loss is equal to $V(L_2^*) - V(L_1^*)$.

convergence towards the stationary solution is possible only if time flows in reverse (cf. Lesovik et al. 2019). From an economic point of view, this result can be rationalized by arguing that when there is only one stationary solution the agreement between the firm and the workers described by the problem in (5) becomes pointless. Indeed, when there is only one resting point in the system of eq.s (2) and (7), the insurance mechanism provided by the implicit contract is undermined. Therefore, the solution of the firm problem is not able to pin down a meaningful out-of-equilibrium dynamics for hours and wages .

Under the condition pointed out by Proposition 2, that is, when there are two distinct stationary solutions given by (L_1^*, w_1^*) and (L_2^*, w_2^*), analytical results are difficult to be derived. However, fixing the value of ρ and relying on a computational package it becomes possible to assess - for different values of A, θ and C - the magnitude of the eigenvalues associated to the Jacobian matrix in (12) - say r_1 and r_2 - for each implied stationary solution.[6] Specifically, setting the value of the discount rate as in Alvarez and Shimer (2011), some sets of numerical solutions are collected in Tables 1, 2 and 3.

A	L_1^*	w_1^*	$r_1(L_1^*, w_1^*)$	$r_2(L_1^*, w_1^*)$	L_2^*	w_2^*	$r_1(L_2^*, w_2^*)$	$r_2(L_2^*, w_2^*)$
1.3	0.351	2.845	$0.025 + 0.039i$	$0.025 - 0.039i$	0.948	1.054	0.091	-0.041
1.4	0.304	3.287	$0.025 + 0.043i$	$0.0250 - 0.043i$	1.095	0.912	0.100	-0.050
1.5	0.271	3.686	$0.025 + 0.046i$	$0.025 - 0.046i$	1.228	0.813	0.107	-0.057
1.6	0.246	4.061	$0.025 + 0.048i$	$0.025 - 0.048i$	1.353	0.738	0.112	-0.062
1.7	0.226	4.421	$0.025 + 0.049i$	$0.025 - 0.049i$	1.473	0.678	0.117	-0.067

Table 1: Numerical solutions for different values of A ($\rho = 0.05$, $\theta = 0.10$, $C = 1$)

θ	L_1^*	w_1^*	$r_1(L_1^*, w_1^*)$	$r_2(L_1^*, w_1^*)$	L_2^*	w_2^*	$r_1(L_2^*, w_2^*)$	$r_2(L_2^*, w_2^*)$
0.08	0.328	3.046	$0.025 + 0.039i$	$0.025 - 0.039i$	1.171	0.853	0.092	-0.042
0.09	0.296	3.368	$0.025 + 0.042i$	$0.025 - 0.042i$	1.203	0.831	0.100	-0.050
0.10	0.271	3.686	$0.025 + 0.046i$	$0.025 - 0.046i$	1.228	0.813	0.107	-0.057
0.11	0.250	4	$0.025 + 0.049i$	$0.025 - 0.049i$	1.250	0.800	0.115	-0.065
0.12	0.231	4.311	$0.025 + 0.052i$	$0.025 - 0.052i$	1.268	0.788	0.122	-0.072

Table 2: Numerical solutions for different values of θ ($\rho = 0.05$, $A = 1.5$, $C = 1$)

C	L_1^*	w_1^*	$r_1(L_1^*, w_1^*)$	$r_2(L_1^*, w_1^*)$	L_2^*	w_2^*	$r_1(L_2^*, w_2^*)$	$r_2(L_2^*, w_2^*)$
0.8	0.206	3.881	$0.025 + 0.049i$	$0.025 - 0.049i$	1.293	0.618	0.116	-0.066
0.9	0.237	3.787	$0.025 + 0.047i$	$0.025 - 0.047i$	1.262	0.713	0.112	-0.062
1.0	0.271	3.686	$0.025 + 0.046i$	$0.025 - 0.046i$	1.228	0.813	0.107	-0.057
1.1	0.307	3.577	$0.025 + 0.044i$	$0.025 - 0.044i$	1.192	0.922	0.103	-0.053
1.2	0.346	3.459	$0.025 + 0.042i$	$0.025 - 0.042i$	1.153	1.040	0.098	-0.048

Table 3: Numerical solutions for different values of C ($\rho = 0.05$, $A = 1.5$, $\theta = 0.10$)

[6] All the MATLAB codes used throughout the paper are available from the authors upon request.

The numerical results in Tables 1, 2 and 3 can be summarized in the following proposition:

Proposition 4: When $A > 2\sqrt{C\rho/(\rho+\theta)}$, the stationary solution (L_1^*, w_1^*) defined in Proposition 2 is an unstable source with complex dynamics whereas (L_2^*, w_2^*) is a saddle point.

Proposition 4 reveals that when the condition for multiple stationary solution is met the steady-state with low equilibrium hours and high equilibrium wage is unstable whereas the steady-state with high equilibrium hours and low equilibrium wage is characterized by saddle-path dynamics. This means that given an initial value for the wage - say $w(0) = \overline{w}_0 > 0$ - there is only one trajectory that satisfies the dynamic system in (10) which converges to (L_2^*, w_2^*) while all the others diverge. In other words, the equilibrium path towards the steady-state with high equilibrium hours and low equilibrium wage is locally determinate, that is, taking the value of $\overline{w}_0$ there is only one a unique value of the initial hours - $L(0)$ - in the neighbourhood of L_2^* that generates a trajectory converging to (L_2^*, w_2^*) whereas all the others diverge. Strictly speaking, the value of $L(0)$ should be selected in order to verify the transversality condition in (8) by placing the system in (12) exactly on the stable branch of the saddle point (L_2^*, w_2^*).

An interesting implication of Proposition 4 is that unless the system rests in (L_1^*, w_1^*), hours and wages tend to converge towards (L_2^*, w_2^*), that is, the allocation preferred by the firm and opposed by its employees. Moreover, everything else being equal, the absolute value of the convergent root towards (L_2^*, w_2^*) is an increasing function (decreasing) of A and θ (C). Using the baseline calibration indicated in the forth row of Tables 1, 2 and 3 and assuming that $w(0)$ is 1% below or above w_2^*, the saddle path dynamics of hours, wage and their product - which is assumed to coincide with workers' consumption stated by the contract - is illustrated in the two panels of Figure 1.

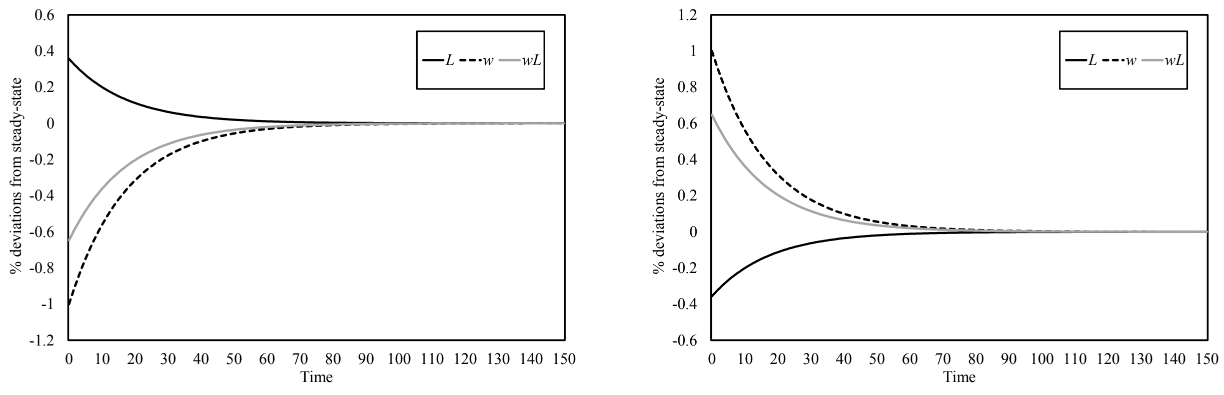

Figure 1: Saddle path adjustments of hours, wages and earnings
$A = 1.5$, $\rho = 0.05$, $\theta = 0.10$, $C = 1$

The two diagrams in Figure 1 show that when the starting level of the wage undershoots (overshoots) its stationary reference by 1% hours overshoot (undershoot) their long-run equilibrium value by 0.35%, whereas earnings undershoot (overshoot) their fixed contractual value by 0.65%. Thereafter, consistently with the micro-econometric tests of the implicit contract theory, wages move counter-cyclically until (L_2^*, w_2^*) is reached. Moreover, given the absence of savings, the whole adjustment process of hours and wages is characterized by a pattern of under- or overconsumption depending on the initial value of the contract wage.

The pattern of hours and wages follows in a straightforward manner from the role played by the wage rate in the model economy under investigation. Indeed, taking into account the insurance scheme provided to workers by the self-enforcing implicit contract, the wage does not play any allocative function but it can be thought as a sort of indemnity that the firm corresponds to its workers with the aim of stabilizing their consumption (cf. Barro, 1977; Hall, 1980). On the side of the firm, large (small) indemnities are profitable only when productivity of labour is high (low) and this happens when the amount of working hours is low (high). On the side of workers, given the targeted stability of consumption, higher (lower) indemnities will be used to buy additional (sell some) leisure - which is assumed to be a normal good - by leading the insured employees to work for a lower (higher) amount of hours. In other words, consistently with wage equations run in the US at the micro level by controlling for labour productivity, higher (lower) wages have only an income effect that leads workers to work less (more) (cf. Beaudry and Dinardo, 1995).

4 The Stochastic model

Now we deal with the more realistic case in which the variable that conveys the realized state of the world and the effectiveness of labour is not constant but it follows the stochastic process in eq. (2). In this case, the firm problem becomes the following:

$$\begin{gathered}\max_{\{L(t)\}_{t=0}^{\infty}} E_0\left[\int_0^{\infty} \exp\left(-\rho t\right)\left(A\left(t\right)L\left(t\right)-\tfrac{1}{2}\left(L\left(t\right)\right)^2-w\left(t\right)L\left(t\right)\right)dt\right] \\ \text{s.to} \\ \dot{w}\left(t\right)=\theta\left(\tfrac{C}{L(t)}-w\left(t\right)\right) \\ \dot{A}\left(t\right)=\kappa\left(\mu_A-A\left(t\right)\right)+\sigma_A\dot{x}\left(t\right) \\ A\left(0\right)=A_0\end{gathered} \tag{16}$$

where $E\left[\cdot\right]$ is the expectation operator whereas $A_0>0$ is the initial value of the state of the world.

The Hamilton-Jacobi-Bellman (HJB) equation for the firm problem is given by

$$\begin{gathered}\rho V\left(A\left(t\right),w\left(t\right)\right)=\max_{L}\left\{A\left(t\right)L\left(t\right)-\tfrac{1}{2}\left(L\left(t\right)\right)^2-w\left(t\right)L\left(t\right)\right. \\ \left.+\theta\left(\tfrac{C}{L(t)}-w\left(t\right)\right)\tfrac{\partial V(A(t),w(t))}{\partial w}+\kappa\left(\mu_A-A\left(t\right)\right)\tfrac{\partial V(A(t),w(t))}{\partial A}+\tfrac{1}{2}\sigma_A^2\tfrac{\partial^2 V(A(t),w(t))}{\partial A^2}\right\}\end{gathered} \tag{17}$$

where $V\left(\cdot\right)$ is the value function.

The FOC for $L(t)$ requires that along an optimal path it must hold that

$$\frac{\partial V\left(A\left(t\right),w\left(t\right)\right)}{\partial w}=\frac{L\left(t\right)^2\left(A\left(t\right)-L\left(t\right)-w\left(t\right)\right)}{C\theta} \tag{18}$$

Moreover, the envelope condition for $w(t)$ is given by

$$\begin{array}{c}(\rho+\theta)\frac{\partial V(A(t),w(t))}{\partial w}=\left(\frac{C}{L(t)}-w(t)\right)\frac{\partial^2 V(A(t),w(t))}{\partial w^2}\\ +\kappa\left(\mu_A-A(t)\right)\frac{\partial^2 V(A(t),w(t))}{\partial A\partial w}+\frac{1}{2}\sigma_A^2\frac{\partial^3 V(A(t),w(t))}{\partial A^2\partial w}-L(t)\end{array} \tag{19}$$

Despite the simplicity of the stochastic process used to describe the evolution of labour effectiveness, analytical results for the dynamics of hours and wages may be difficult to find. Nevertheless, the solution of the stochastic model can be retrieved by using numerical techniques aimed at approximating the value function over a given grid (cf. Kushner and Dupuis, 1992).[7]

4.1 Calibration and simulation results

The stochastic model is simulated in order to match the volatility of the log-deviations of US GDP from its long-run level as reported by Ravn and Simonelli (2007). In other words, we calibrate the model with the aim of replicating the volatility of the observed output fluctuations. To this end, the baseline calibration indicated in the forth row of Tables 1, 2 and 3 is integrated by calibrating the stochastic process in eq. (2) in the following manner. First, the long-run mean of the stochastic process that conveys the effectiveness of production (μ_A) is set at the same value exploited for the deterministic simulations in Figure 1. Second, the speed of mean reversion of the effectiveness of production (κ) is fixed at the value of the convergent root of implied by the baseline calibration of the deterministic model. Moreover, the volatility of the effectiveness of production (σ_A) is tuned in order to achieve the targeted value of the standard deviation of output.[8] The whole set of parameters, their description and the respective values are collected in Table 4.

Parameter	Description	Value
C	*Long-run consumption*	1.000000
ρ	*Discount rate*	0.050000
θ	*Attrition of the contract wage*	0.100000
μ_A	*Long-run productivity*	1.500000
κ	*Attrition of productivity*	0.057000
σ_A	*Standard deviation of productivity*	0.004225

Table 4: Calibration

Given the parameters' value in Table 4, the theoretical values implied by the model economy are retrieved by following the typical steps followed in business cycles contributions (cf. Shimer, 2005). Specifically, we first generate $1,200$ theoretical observations. Throwing away the first $1,000$ in order to mitigate the butterfly effect, we remain with 200 observations that represent

[7]As we explain in Appendix, a promising method for simulating the dynamic implicit-contract model with stochastic labour effectiveness is the one grounded on a Markov decision chain approximation.

[8]The calibration is completed by fixing $w_0 = 0.81$, $A_0 = 1.51$ and setting the time-step of simulation to 0.004.

the corresponding quarterly figures of the typical 50-year horizon covered by business cycle analysis. For each variable of interest, we take the standard deviation and the correlation matrix of the log deviations from the corresponding deterministic long-run reference. Thereafter, such a procedure is repeated for 10,000 times and theoretical values are obtained by averaging the outcomes of each replication. Defining $\overline{z}$ as $\ln z - \ln z^*$, where z^* is the stable stationary solution for the variable z, the simulation results for a set of selected variables are collected in Table 5 (observed values in parenthesis).

Variable		$\overline{Y}$	$\overline{wL}$	$\overline{L}$	$\overline{w}$
Standard deviation (%)		1.56 (1.56)	0.57 (1.01)	0.92 (0.51)	0.69 (0.86)
Correlation matrix	$\overline{Y}$	1	0.63	0.94 (0.67)	−0.71 (0.18)
	$\overline{wL}$	–	1	0.66	−0.04
	$\overline{L}$	–	–	1	−0.76 (0.01)
	$\overline{w}$	–	–	–	1

Table 5: Simulation results

The figures in Table 5 suggest the following broad conclusions. First, the stochastic model understates the volatility of labour earnings and wages but it overstates the one of working hours. According to simulated figures, earnings should be the variable with the smaller volatility while in real data the lowest dispersion is instead observed for hours. If we interpret earnings as a measure of consumption, then the figure of volatility is still understated though to a lower extent; indeed, the observed standard deviation of consumption amounts to 0.86% which is definitely higher than 0.57%. An explanation for this pattern is that our theoretical framework does not account for the consumption of unemployed workers which is usually more volatile than the consumption of the employed (cf. Pissarides, 2004). Second, contrary to what is shown by the deterministic model, the stochastic model displays a sound degree of real-wage stickiness; indeed, the standard deviation of simulated wages is more than double with respect to the one of output. In comparison with actual data, however, our theoretical model tends to exacerbate the cyclical correlation of hours with respect to output. Moreover, the stochastic model replicates in a strong manner the counter-cyclicality of wages that also characterizes the saddle-path trajectories of the deterministic model. Obviously, this means that the insurance scheme implied the dynamics of hours and wages is unable to explain the mild pro-cyclicality of wages observed at the macro level (cf. Calmès, 2007). An example of a typical trajectory of hours, wages and labour earnings is illustrated in Figure 2.

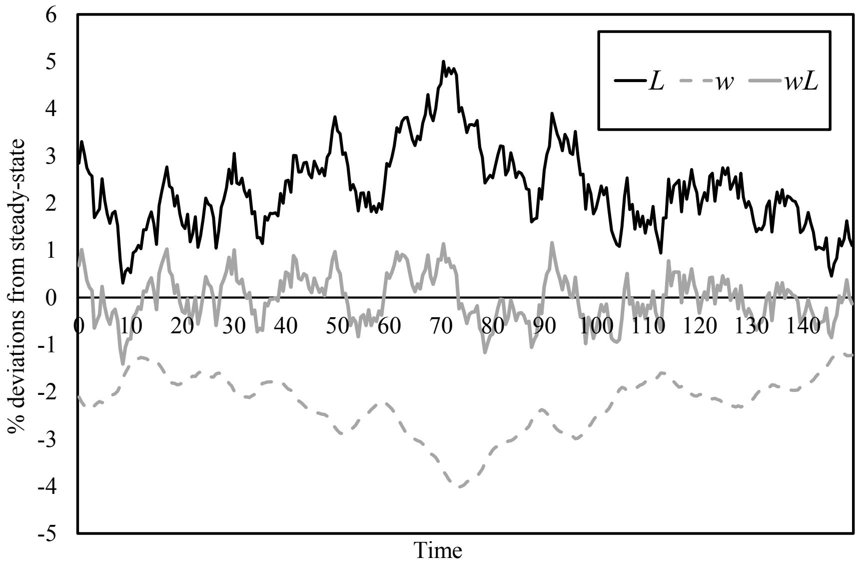

Figure 2: Stochastic adjustments of hours, wages and earnings

The diagram in Figure 2 clearly shows the pronounced consumption smoothing operated by the dynamic implicit contract via the dynamics of labour earnings as well as the counter-cyclical behaviour of wages; indeed, working hours (wages) are always above (below) their stable long-run references. Such a pattern reveals the existence of a strong amplification mechanism of productivity shocks inside the stochastic model coming from the rigidity of wages. Although the negative correlation between hours and wage appear at odds with the available macro evidence, that kind of dynamic behaviour is a direct consequence of the insurance scheme described above and is also consistent with the empirical tests of the implicit contract theory carried out with micro data on hours and wages even outside the US (cf. Bellou and Kaymak, 2012).

5 Concluding remarks

In this paper, we develop a dynamic implicit-contract model grounded on optimal control. Specifically, we explore the out-of-equilibrium dynamics of working hours and wages in a model economy where a risk-neutral representative firm and its risk-averse workers have agree upon a self-enforcing implicit contract that is assumed to smooth long-run real labour incomes and consumption. More in details, we develop a theoretical framework in which the firm inter-temporally sets its optimal level of labour utilization by taking into account that the implied wage bill tends to adjusting in the direction of a fixed level that seeks to stabilizing workers' equilibrium consumption (cf. Romer, 2019, Chapter 11).

Ignoring uncertainty in labour productivity reveals that our theoretical setting may have one, two or none stationary solution. However, the out-of-equilibrium dynamics of the deterministic economy can be assessed only in the two-solution case and it reveals that wages tend to moving in the opposite direction with respect to working hours by converging towards the allocation in which firm's profit is high and workers' utility is low. This result corroborates the micro-econometric evidence on the implicit contract theory obtained by regressing wage on hours by controlling for productivity (cf. Beaudry and DiNardo, 1995). Moreover, when the initial value of the contract wage falls above (below) its long-run equilibrium value, the pattern of workers' consumption is characterized by overconsumption (underconsumption).

Adding uncertainty in labour productivity with the aim of replicating the magnitude of observed output fluctuations has the potential to miming the real wage stickiness conveyed by macro data (cf. Ravn and Simonelli, 20017). However, the insurance mechanism provided by the dynamic implicit contract understates the volatility of labour earnings and confirms the counter-cyclicality of wages observed in micro data in a number of countries (cf. Bellou and Kaymak, 2012).

The failure of the model to predicting a pro-cyclical pattern of wages is easily attributable to the lack of adjustments on the extensive margin. If positive shocks to the effectiveness of labour lead the firm to hire workers and the path of contract wages is given, the marginal productivity of working hours does not necessarily moves in the same direction of the effectiveness of labour because not only its vertical intercept but also its shape will be affected by the level of employment. Obviously, this may open the door to a positive co-movement of hours, employment and wages as observed in real data. The implied extension of the model is left to further developments.

Appendix: The fixed level of consumption

Consider the production function in eq. (1). For sake of simplicity, suppose that there only two states of the world. This means that the productivity parameter A can take only two values so that it is equal to A_1 (A_2) with probability p $(1-p)$. Moreover, assume that workers-consumers are endowed with a utility function separable in consumption and leisure where both components - in analogy with the production possibilities of the firm - are given by distinct quadratic expressions. Formally speaking, this amounts to positing that the utility function of workers can be written as

$$U(C_i, L_i) = ZC_i - \frac{1}{2}C_i^2 - VL_i + \frac{1}{2}L_i^2 \qquad Z > 0,\, V > 0,\, i = 1, 2 \tag{A1}$$

where C_i is consumption, L_i is labour provision measured in hours whereas Z and V are positive parameters.

In a time-less economy described by the production function in eq. (1) and the preferences in (A1), the optimal implicit contract that smooths workers' consumption in all the possible states of the world is found through the solution of the following problem:

$$\max_{\{C_i, L_i\}_{i=1,2}} p\left(A_1L_1 - \frac{1}{2}L_1^2 - C_1\right) + (1-p)\left(A_2L_2 - \frac{1}{2}L_2^2 - C_2\right) \tag{A2}$$

s.to

$$p\left(ZC_1 - \frac{1}{2}C_1^2 - VL_1 + \frac{1}{2}L_1^2\right) + (1-p)\left(ZC_2 - \frac{1}{2}C_2^2 - VL_2 + \frac{1}{2}L_2^2\right) \geqslant u_0 \qquad \text{(A3)}$$

where u_0 is a fixed level of the fallback utility level of workers (cf. Romer, 2019, Chapter 11).

The FOCs of the problem in (A2) – (A3) are given by

$$-1 + \lambda (Z - C_i) = 0 \qquad i = 1, 2 \qquad \text{(A4)}$$

$$A_i - L_i - \lambda (V - L_i) = 0 \qquad i = 1, 2 \qquad \text{(A5)}$$

where λ is the Lagrange multiplier associated to the participation constraint in (A3).

The FOCs in (A4) implies that consumption is constant in all the states of the world so that

$$C_i = C^{**} > 0 \qquad i = 1, 2 \qquad \text{(A6)}$$

Plugging the FOCs in (A4) in (A4) into the FOCs in (A5) by taking into account the result in (A6) leads to

$$L_i^{**} = \frac{A_i (Z - C^{**}) - V}{Z - C^{**} - 1} \qquad i = 1, 2 \qquad \text{(A7)}$$

The expression in (A7) reveals that in each state of the world contract employment is found by equalizing the marginal productivity of labour to the marginal rate of substitution between consumption and leisure pinned down by the fixed level of C. In other words, under the optimal contract the actual level of employment is given by the intersection of the conventional labour demand schedule and a constrained labour supply in which the possibilities of substitution between consumption and leisure are bound by the fact that in each state of world workers have to consume exactly the amount C^{**}.

Plugging the results in (A6) and (A7) into (A3) reveals that C^{**} has to be consistent with the following equation:

$$ZC^{**} + \frac{(L_2^{**})^2 - (C^{**})^2}{2} - VL_2^* + p\left(\frac{V(Z - C^{**})(A_2 - A_1)}{Z - C^{**} - 1} + \frac{(L_1^{**})^2 - (L_2^{**})^2}{2}\right) = u_0 \qquad \text{(A8)}$$

For reasonable values of A_1, A_2, Z, V, and p, the expression on left-hand-side (LHS) of (A8) is monotonically decreasing in C. Consequently, as shown in figure A1, given a positive value of u_0 there exists an unique meaningful value of C^{**}.

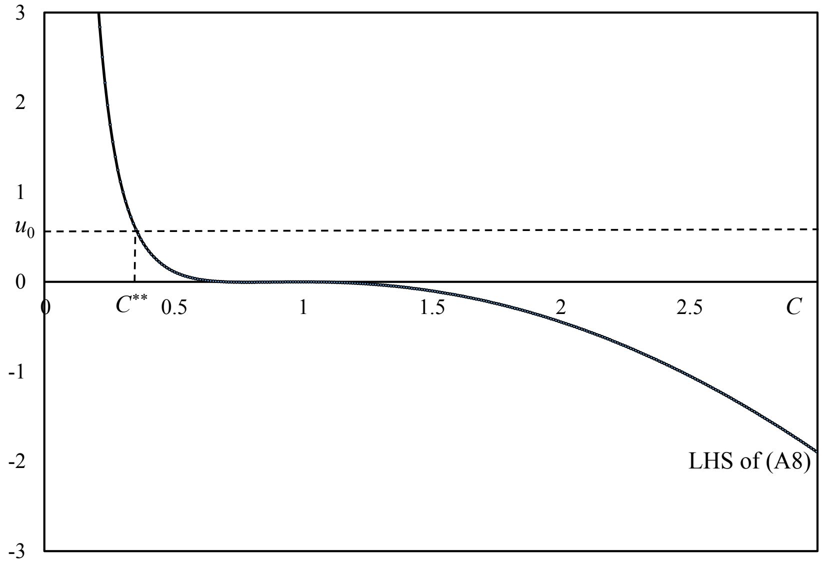

Figure A1: The determination of C^{**}
$A_1 = 1,\ A_2 = 2,\ Z = V = 1,\ p = u_0 = 1/2$

Appendix: An alternative for the dynamics of contract wages

A reasonable alternative for the real wage dynamics fixed by the contract is the one according to which the real wage increases (decreases) when the real wage bill is below (above) the long-run level of consumption. Formally, speaking a sensible alternative for the differential equation in (2) is given by

$$\dot{w}(t) = \theta\left(C - w(t)L(t)\right) \tag{B1}$$

In this case, the solution of the firm problem leads to following employment dynamics:

$$\dot{L}(t) = \frac{\theta\left(A - L(t)\right)\left(w(t)L(t) - C\right) - w(t)\left(A\rho + L(t)\left(A\theta - \rho\right) - \theta\left(L(t)\right)^2 - \rho w(t)\right)}{w(t)} \tag{B2}$$

The differential equations in (B1) and (B2) imply that steady-state level of hours, that is L^*, is consistent with a cubic continuous expression defined as

$$\Psi\left(L^*\right) \equiv \theta\left(L^*\right)^3 - \left(L^*\right)^2\left(A\theta - \rho\right) - A\rho L^* + \rho C = 0 \tag{B3}$$

Straightforward differentiation revels that the function $\Psi(\cdot)$ has two critical points given by

$$\overline{L} \equiv \frac{A\theta - \rho - \sqrt{A\theta\left(A\theta + \rho\right) + \rho^2}}{3\theta} \text{ and } \overline{\overline{L}} \equiv \frac{A\theta - \rho + \sqrt{A\theta\left(A\theta + \rho\right) + \rho^2}}{3\theta} \tag{B4}$$

Since $\Psi(0) = \rho C$ and $\lim_{L^* \to +\infty(-\infty)} \Psi(L^*) = +\infty\,(-\infty)$, $\overline{L}$ $(\overline{\overline{L}})$ is a maximum (minimum) for $\Psi(\cdot)$. Consequently, it becomes possible to stating the following propositions:

Proposition B1: When $\Psi\left(\overline{\overline{L}}\right) = 0$, $\left(\overline{\overline{L}}, C/\overline{\overline{L}}\right)$ is the only stationary solution to the system of differential equations given by (B1) and (B2).

Proposition B2: When $\Psi\left(\overline{\overline{L}}\right) < 0$, there are two distinct stationary solutions to the system of differential equations given by (B1) and (B2), namely $(L_1^*, C/L_1^*)$ and $(L_2^*, C/L_2^*)$ such that $0 < L_1^* < \overline{\overline{L}}$ and $L_2^* > \overline{\overline{L}}$.

Proposition B3: When $\Psi\left(\overline{\overline{L}}\right) > 0$, there are no (real) stationary solutions to the system of differential equations given by (B1) and (B2).

Propositions B1-B3 are qualitatively similar to Propositions $1-3$. Moreover, using a computational package is possible to show that the set of stationary solutions catalogued by Propositions B1 and B2 have the same dynamic properties of the stationary solutions analyzed in the main text. Specifically, for $\left(\overline{\overline{L}}, C/\overline{\overline{L}}\right)$ is not possible to retrieving local dynamics, $(L_1^*, C/L_1^*)$ is an unstable source whereas $(L_2^*, C/L_2^*)$ is a saddle point. Further details are available from the authors upon request.

Appendix: Simulating the stochastic model with a Markov decision chain approximation

Here we examine a mathematical tool that allows to solving numerically the stochastic optimal control problem outlined in Section 3. The approach implemented by the tool is described by Krawczyk and Windsor (1997) and here we provide the principal ideas underlying its solution method.

The first step is the discretization of the state-equation system using the Euler-Maruyama approximation scheme (cf. Kloeden and Platen, 1992). Consider the following general continuous-time form:

$$d\mathbf{X}(t) = f\left(\mathbf{X}(t), \mathbf{u}(t), t\right) dt + b\left(\mathbf{X}(t), \mathbf{u}(t), t\right) d\mathbf{W}(t) \tag{C4}$$

where $\mathbf{X}$ is the vector of state variables, $\mathbf{u}(t)$ is the vector of control variables whereas $\mathbf{W}(t)$ is a Wiener process.

According to the Euler-Maruyama scheme, the approximation of (C4) in N partitions is given by

$$\mathbf{Y} = \{\mathbf{Y}_l, l \in \mathbb{N}, 0 \leq l \leq N\} \tag{C5}$$

The expression in (C5) has to be consistent with the following expression:

$$\mathbf{Y}_{l+1} = \mathbf{Y}_l + f\left(\mathbf{Y}_l, \mathbf{u}_l, \tau_l\right)(\tau_{l+1} - \tau_l) + b\left(\mathbf{Y}_l, \mathbf{u}_l, \tau_l\right)\left(\mathbf{W}_{\tau_{l+1}} - \mathbf{W}_{\tau_l}\right) \tag{C6}$$

where $l = 0, \ldots, N-1$ whereas the initial seed equals to $\mathbf{Y}_0 = \mathbf{X}(0)$.

Thereafter, in order to determine the Markov decision process, we have to define a discrete state space, transition probabilities for each state, as well as a reward function associated with each transition. The discrete state space for stage l is denoted by $\overline{X}_l$ whereas the extreme values of the state grid are given by $\overline{U}_l = \max\left\{\overline{X}_l\right\}$ and $\overline{L}_l = \min\left\{\overline{X}_l\right\}$. Consequently, a point $x \in X$ is in the grid $\overline{X}_l$, if and only if $\overline{L}_l \leq x \leq \overline{U}_l$. Moreover, the set of the discrete state spaces for all stages, formally speaking $\left\{\overline{X}_l\right\}_{l=0}^{N}$, is denoted by $\overline{X}$. Heuristically, the adopted numerical scheme is able to approximate a generic point of X at stage l by the points of $\overline{X}_l$ which are adjacent to it.

Having defined the discrete state space, we now move to the definition of the transition probabilities. Consider the stochastic process in eq. (C5), i.e., $\mathbf{Y} = \{\mathbf{Y}_l, l = 0, \ldots, N\}$, where $\mathbf{Y}_l$ is defined by (C6). This process, although defined at discrete times, can take any real value.

For a given control sequence $\mathbf{u}_l$ and an equidistant discretization time-steps, we can re-write the iterative scheme of (C6) in the following abbreviated form:

$$\mathbf{Y}_{l+1} = \mathbf{Y}_l + \delta f_l + b_l \Delta\mathbf{W}_l \tag{C7}$$

where $f_l = f\left(\mathbf{Y}_l, \mathbf{u}_l, \tau_l\right)$, $b_l = b\left(\mathbf{Y}_l, \mathbf{u}_l, \tau_l\right)$ whereas $\Delta\mathbf{W}_l = \mathbf{W}_{\tau_{l+1}} - \mathbf{W}_{\tau_l}$.

Suppose that we are at time τ_l, so that $\mathbf{Y}_l = \overline{\mathbf{Y}}_l \in \overline{X}_l$. In a deterministic context (i.e., whenever $\Delta\mathbf{W}_l = 0$, $l = 0, \ldots, N-1$), for a given control value $\mathbf{u}_l$, the process moves to $\mathbf{Y}_{l+1}$, according to eq. (C7). Consequently,

$$\mathbf{Y}_{l+1} = \overline{\mathbf{Y}}_l + \delta f_l \tag{C8}$$

If $\mathbf{Y}_{l+1}$ has only one state adjacent to it, then the transition probability from $\mathbf{Y}_l$ is equal to 1. By contrast, if there is a pair of states adjacent to $\mathbf{Y}_{l+1}$, called $\left(\overline{\mathbf{Y}}_{l+1}^{\ominus}, \overline{\mathbf{Y}}_{l+1}^{\oplus}\right)$, such that $h_l = \overline{\mathbf{Y}}_{l+1}^{\oplus} - \overline{\mathbf{Y}}_{l+1}^{\ominus} > 0$, the transition probabilities determined according to an inverse distance method. Formally speaking, we have

$$\begin{aligned} p\left(\overline{\mathbf{Y}}_l, \overline{\mathbf{Y}}_{l+1}^{\oplus}|\mathbf{u}_l\right) &= \tfrac{\mathbf{Y}_{l+1} - \overline{\mathbf{Y}}_{l+1}^{\ominus}}{h_l} \\ p\left(\overline{\mathbf{Y}}_l, \overline{\mathbf{Y}}_{l+1}^{\ominus}|\mathbf{u}_l\right) &= \tfrac{\overline{\mathbf{Y}}_{l+1}^{\oplus} - \mathbf{Y}_{l+1}}{h_l} \end{aligned} \tag{C9}$$

In a stochastic context, a Gaussian noise is present in (C7) and, consequently, $\mathbf{Y}_{l+1}$ is no more deterministic. In this case, we can use a partition of the realizations of the Gaussian process $\Delta\mathbf{W}_l$ into M steps. If we choose $M = 3$ and we use these intervals: $\left(-\infty, -\sqrt{\delta}\right)$, $\left(-\sqrt{\delta}, +\sqrt{\delta}\right)$, $\left(+\sqrt{\delta}, +\infty\right)$, where $\sqrt{\delta}$ is the standard deviation of $\Delta\mathbf{W}_l$, then we can compute the expected values of the noise by using the following expression:

$$\omega = \frac{\sqrt{2\delta}}{\sqrt{\pi e}\left(1 - erf\left(\frac{1}{\sqrt{2}}\right)\right)} \tag{C10}$$

where erf is the standard error function defined by $erf(t) = 2/\sqrt{\pi}\int_0^t e^{-t^2}dt$.

The transition probabilities for an approximated situation in which the process $\mathbf{Y}$ is perturbed by the discretely valued noise ω_l are defined by:

$$\begin{aligned} P(\omega_l = -\omega) &= p_- \\ P(\omega_l = 0) &= p_0 \\ P(\omega_l = +\omega) &= p_+ \end{aligned} \tag{C11}$$

As a result, if $\mathbf{Y}_{l+1}$ is obtained by (C7) and there is a single adjacent state, then the process reaches $l+1$ with the following probabilities:

$$\begin{aligned} \mathbf{Y}^-_{l+1} &= \mathbf{Y}_{l+1} + b_l\omega_- \text{ with probability } p_- \\ \mathbf{Y}^0_{l+1} &= \mathbf{Y}_{l+1} \text{ with probability } p_0 \\ \mathbf{Y}^+_{l+1} &= \mathbf{Y}_{l+1} + b_l\omega_+ \text{ with probability } p_+ \end{aligned} \tag{C12}$$

By contrast, if there are two adjacent states, then it is reasonable to apply the inverse distance method as in (C9) weighted by the proper probabilities defined in (C12). For instance, if we consider $\mathbf{Y}^-_{l+1}$ with the two adjacent states $\overline{\mathbf{Y}}^{-\ominus}_{l+1}$ and $\overline{\mathbf{Y}}^{-\oplus}_{l+1}$, then the transition probabilities are given by

$$\begin{aligned} p\left(\overline{\mathbf{Y}}_l, \overline{\mathbf{Y}}^{-\oplus}_{l+1}|\mathbf{u}_l\right) &= p_-\frac{\mathbf{Y}_{l+1}-\overline{\mathbf{Y}}^{-\ominus}_{l+1}}{h_l} \\ p\left(\overline{\mathbf{Y}}_l, \overline{\mathbf{Y}}^{-\ominus}_{l+1}|\mathbf{u}_l\right) &= p_-\frac{\overline{\mathbf{Y}}^{-\oplus}_{l+1}-\mathbf{Y}_{l+1}}{h_l} \end{aligned} \tag{C13}$$

where $h_l = \overline{\mathbf{Y}}^{-\oplus}_{l+1} - \overline{\mathbf{Y}}^{-\ominus}_{l+1} > 0$.

The next phase is to assign the performance function at every transitions of the Markov chain. The objective function to maximize must be the discretised version of the original performance function J on the allowable controls, i.e., $\max_{\mathbf{u}} J(0, \overline{x}_0; \mathbf{u})$ subject to eq. (C7). This completes the description of the tool.

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