Supply and Demand Function Equilibrium: Trade in a Network of Superstar Firms*

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Abstract

This paper studies how input-output connections among firms determine the distribution and the welfare impact of market power in a production network. Firms compete by choosing supply and demand functions relating quantities to prices. In this way, firms' ability to affect prices, total surplus, and its distribution are endogenous objects and are determined in equilibrium by technology, the number of competitors, and the network structure. In particular, firms take strategically into account their position in the network, and have market power on both input and output markets, to an extent that is determined in equilibrium. In models in which firms do not take into account their position in the network, I show that that market power is weaker, and the final price larger. As a consequence, in such models the aggregate welfare impact of oligopolies is underestimated, and some vertical mergers might be evaluated as welfare-improving when they are not. Assuming one-sided market power in either output or input markets can reverse the ranking of market power among sectors, as measured for example by the welfare impact of a horizontal merger. An equilibrium always exists for any network under a technology that yields quadratic profit functions, and I provide an algorithm to compute it, that is computationally feasible on realistic networks. Finally,

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horizontal mergers (in absence of synergies) always increase the final price, despite the presence of countervailing power.

1 Introduction

Production of goods in modern economies typically features long and interconnected supply chains. Moreover, many sectors are characterized by *superstar firms*, whose size is very large relative to their sector or even the whole economy.[1] How are prices formed in this setting? How is surplus split? How efficient is the process? This paper provides a strategic non-cooperative model of large firms interacting in an input-output network consisting of many specific supply-customer relationships. The model satisfies two requirements:

R1. Symmetric market power : all firms have market power over both input and output goods, and no prices or quantities are taken as given.

R2. Global strategic interactions : firms strategically take into account their position in the network.

To be concrete, consider a competition authority in charge of evaluating merger proposals. Since evaluation takes time and effort, the authority wants to decide on which sectors to focus on.[2] In order to do this, we must be sure not to build into our models assumption that privilege some sectors/firms with respect to others. For example, Section 7 shows that in some simple sequential models that have been customarily used ad-hoc differences in the order of moves changes the answer completely. Hence, the importance of requirement R1.

The quantification of the distortions that may arise due to market power has attracted a lot of attention recently, with many scholars arguing that competition is in fact decreasing and market power on the rise.[3] In particular, Baqaee and Farhi (2017b) find that taking the input-output connections into account can dramatically increase the size of the implied misallocation in the economy. This paper shows that, if firms take strategically into account their position in the supply chain, the welfare loss is even larger. Hence, requirement R2 is important to be able to correctly evaluate welfare losses.

The novelty of my approach is to incorporate both requirement R1 and R2. Firms' simultaneously commit to supply and demand functions (a uniform price double auction), a methodology first introduced in Grossman (1981) and Klemperer and Meyer (1989). In this context, firms' market

[1] In the terminology of Autor et al. (2020).

[2] Indeed, this is a common issue: for example, in the USA, legislation requires firms to report merger proposals to the relevant authorities, but only for mergers such that the assets of the firms involved lie above some pre-specified thresholds (see e.g. Wollmann (2019))

[3] De Loecker et al. (2020), Gutiérrez and Philippon (2016), Gutiérrez and Philippon (2017).

power is directly connected with the slope of the supply and demand schedule used, with a mechanism similar to the usual inverse elasticity rule in monopoly pricing. The slopes are endogenous and, by treating input and output goods symmetrically, market power is hence solely determined by network position, competition, and technology. This way, there is no need to introduce asymmetries in the timing of firms' choices and treatment of inputs with respect to outputs. The split of the surplus is also endogenous, and there is no need to introduce parameters connected to bargaining. What allows tractability, in a similar way to most models using supply and demand schedules, are a quadratic functional form for the technology and uncertainty in some (cost) parameters. The quadratic functional form yields linearity of schedules in equilibrium, while uncertainty pins down the best replies uniquely.

Theorems 1, 2 and 3 are the main results of the paper.

Theorem 1 provides an existence result for a Supply and Demand Function Equilibrium in linear strategies in any network. This approach does does not need the assumption that the network is acyclic, as for example the sequential models do.[4] The proof relies on the *strategic complementarity* property of the game: the best reply to a steeper supply curve is a steeper supply curve, where "slopes", being matrices of coefficients, are ordered in the positive semidefinite sense. So in this context, when a firm has larger market power, every other firm has more market power in turn.

Theorem 2 shows that in this setting mergers always increase market power. If, in addition, there is a single aggregate final good, mergers increase the final price. Strategic complementarities are key again: the merged firm will face less competition and so choose a flatter schedule, triggering complementary responses from direct competitors, suppliers and customers, and all firms connected through the network.

Theorem 3 shows that ignoring global strategic considerations (requirement R2) leads to less market power: in a similar model of competition in supply and demand functions, in which firms ignore the rest of the network, market power distortions are smaller. This is because if a firm does not internalize some reactions in the network, this amounts to the firm perceiving a larger elasticity of demand.

The rest of the paper is organized as follows. The next paragraphs describe the related literature. Section 2 defines the model in full generality and then explains the parametric assumptions needed for the analysis. Section 3 illustrates the solution and the existence theorem. Section 7 presents some of the benchmark models discussed above and clarifying the differ-

[4]The relevance of cycles in real production networks is not yet very clear, but on strict terms, they are not acyclic. For example, Tintelnot et al. (2018) estimates that no more than 23% of the links in the Belgian firm-to-firm production network violate acyclicity. This might justify assuming acyclicity as a first approximation, but is a number distant from zero.

ences with my approach. Section 4 illustrates the welfare impact of mergers in Theorem 2. Section 5 presents the local version of the model and Theorem 3. Section 6 explains how it is possible to solve the model numerically. Section 8 concludes.

Related literature This paper contributes to three lines of literature: the literature on competition in supply and demand functions, the literature on production networks, and the literature on market power in networked markets.

The use of supply schedules as choice variables in the analysis of oligopoly was introduced in Grossman (1981), and in its modern form by Klemperer and Meyer (1989). These studies feature market power on one side of the market only, as typical in oligopoly models.[5] Vives (2011) studies the case of asymmetric information. Akgün (2004) studies mergers among firms competing in supply functions, without the network dimension, finding that mergers are always welfare-decreasing. My results show that some of the mechanisms extend not only to bilateral trade but to trade in any network. Among the papers that have dealt with the problem of bilateral oligopoly, allowing for market power on both sides of the market, Hendricks and McAfee (2010) is a model of bilateral oligopoly where players compete through choosing a capacity parameter: the elasticity of the demand and supply schedules is a given. My contribution with respect to them is a setting in which the elasticities (slopes) of demand and supply are themselves endogenous. Weretka (2011) attacks the problem constraining the schedules to be linear (instead of getting this as an equilibrium result), thus gaining traction in the analysis for general functional forms.

The use of supply and demand schedules is common also in the finance microstructure literature and in the literature on multi-unit auctions. In finance it was introduced and popularized by Kyle (1989). From a technical point of view, the closest paper to mine is Malamud and Rostek (2017), which studies trade in interconnected financial markets: some of their results are formally similar to the "local" version of the model discussed in Section 5. Ausubel et al. (2014) compare uniform price auctions with pay-as-bid auctions and hybrid approaches.

It is convenient to divide the literature on market power in networks in four parts: sequential models, local competition (sector-level), matching, and bargaining. All differ from my approach, by departing from Requirements 1 and 2. Sequential models of supply chains have been studied in the context of double marginalization by Spengler (1950), in the context of vertical foreclosure by Salinger (1988), Ordover et al. (1990). Recently they

[5]These techniques are customarily used in the study of competition in electricity markets, since Green and Newbery (1992). For a recent contribution, see Delbono and Lambertini (2018).

have been studied in Hinnosaar (2019) (price setting), Federgruen and Hu (2016) (quantity setting), Kotowski and Leister (2019) (sequential auctions). Carvalho et al. (2020) build a tractable model to identify "bottlenecks" in real production network data. In their terminology bottlenecks are those firms that, if removed, would increase welfare. This mechanism is crucially different from mine, because in their model links have exogenous capacity constraints, and removing a firm removes the capacity constraint. By contrast, in my approach the amount of trade is the result of the balance of market powers, and removing a firm leads always to an increase in market power.

Papers where competition is at the sector level assume that either the markup is an exogenously given *wedge* between prices and marginal costs, such as Baqaee and Farhi (2017b), Huremovic and Vega-Redondo (2016)); or is determined by oligopolistic competition *at the sector level*: Grassi (2017), De Bruyne et al. (2019), Baqaee (2018). My results suggest that care has to be taken in using this models to analyze welfare: limiting strategic interaction at the sector level might make oligopolies less inefficient. Acemoglu and Tahbaz-Salehi (2020) build a model where prices are formed through a link-level bargaining process. This means that relative market power, though affected by the network, will be crucially affected by the choice of bargaining parameters. This means that, e.g., the relative market power across sectors (hence the relative importance of mergers) is crucially affected by these exogenous parameters: in my approach, the split of the surplus is instead endogenous and depends only on the technology parameters and the connections. Example 13 illustrates this point.

Also relevant are models that employ cooperative tools, such as stability and matching. The literature started by Hatfield et al. (2013) and recent contributions are Fleiner et al. (2018) and Fleiner et al. (2019). They consider indivisible goods and firms that are price-takers. Fleiner et al. (2019) studies the model in presence of frictions, that are exogenously given through the utility functions, and not the result of the strategic use of market power.

Some papers study the interconnection of final markets of different firms, without analyzing the input-output dimension. In this category fall Bimpikis et al. (2019), Pellegrino (2019), Chen and Elliott (2019).

My paper is also connected to an older line of literature, called "general oligopolistic competition", studying how to represent a full economy with interconnected trades as a game (for a review see Bonanno (1990)).[6] My

[6]The closer in spirit is Nikaido (2015), who uses the market clearing conditions to back up quantities as functions of prices, but his method is limited to Leontief technology, and Benassy (1988) which defines an objective demand by means of a fixprice equilibrium, thus not limiting himself to constant returns technology, but as a drawback having to contemplate a rationing rule, and losing a lot in terms of tractability. These methods are the analogous in their setting of the residual demand in 3.2. Other important contributions are Dierker and Grodal (1986), Gabszewicz and Vial (1972), and Marschak and Selten

contribution is to achieve a fully strategic model of the production side of the economy through the use of competition in supply and demand functions. A recent paper expanding on these themes is by Azar and Vives (2018), that develop a model of firms having market power on output and input markets, but without the input-output dimension.

2 The Model

In this section I first define the model in full generality, that is without making parametric assumptions on the technology and the consumer utility, to clarify the generality of the setting. In paragraph 2.2 I discuss the parametric assumptions needed for the subsequent analysis.

2.1 General setting

Firms and Production Network There are N firms and M goods: their sets are respectively denoted $\mathcal{N}$ and $\mathcal{M}$. Each good might be produced by more firms, so $N \geq M$. I write $i \to g$ if firm i produces good g. Each firm needs as inputs the goods produced by a subset $\mathcal{N}_i^{in}$ of other firms, and sells its outputs to a subset of firms $\mathcal{N}_i^{out}$. $\mathcal{N} = \mathcal{N}_i^{out} \cup \mathcal{N}_i^{in}$ is the *neighborhood* of i. Firms and their connections define a weighted directed graph $\mathcal{G} = (\mathcal{N}, E)$ which is the *input output network* of this economy.

I denote d_i^{out} the *out-degree* and d_i^{in} as the *in-degree* of firm i. Firms are connected if one is a customer of the other. E is the set of existing connections, $E \subseteq N \times N$.

Inputs of firm i are $q_{ij}, j = 1, \dots d_i^{in}$ and outputs $q_{ki}, k = 1, \dots d_i^{out}$. I denote the transformation function available to i as Φ_i. This is a function of the input and output quantities, and also of a stochastic parameter ε_i that has the role of a technological shock. The production possibility set of firm i is $\{(q_{ki})_k, (q_{ij})_j, \ell_i \mid \Phi_i((q_{ki})_k, (q_{ij})_j, \varepsilon_i) = 0\}$. Distinct firms can produce the same good or differentiated goods.

Consumers Consumers are a continuum and identical, so that there is a representative consumer.[7] The labor market is assumed competitive, that in particular means firms will have no power over the wage. Hence the wage plays no role, and so we are going to assume that the labor is the numeraire good, and normalize it to 1 throughout. Similarly to the firms, I am going to assume that the consumer utility depends on stochastic parameters $\varepsilon_c = (\varepsilon_{i,c})_i$, one for each good consumed: $U(c, L, \varepsilon_{i,c})$. The joint distribution of

(2012).

[7]In particular, it is assumed that each infinitesimal consumer owns identical shares of all the firms so that we avoid the difficulties uncovered by Dierker and Grodhal: see the Introduction.

$\varepsilon = ((\varepsilon_i)_i, (\varepsilon_{i,c})_i)$ as F. Denote the demand for good i derived by U as $D_{ci}(p_i, \varepsilon_c)$.

Notation I write p_i^{out} for the vector of all prices of ouputs of sector i, and p_i^{in} for the vector of input prices, and $p_i = \begin{pmatrix} p_i^{out} \\ -p_i^{in} \end{pmatrix}$. u_i^{out} denotes a vector of ones of lenght d_i^{out}, while u_i^{in} a corresponding vector of ones of length d_i^{in}, and $\tilde{u}_i = \begin{pmatrix} u_i^{out} \\ -\omega_i \end{pmatrix}$. Similarly, I_i is the identity matrix of size $d_i^{out} + d_i^{in}$, while I_i^{in} and I_i^{out} have respectively size d_i^{in} and d_i^{out}.

Unless specified differently, the inequality $B \geq C$ when B and C are matrices denotes the *positive semidefinite* (Löwner) ordering. That is: $B \geq C$ if and only if $B - C$ is positive semidefinite.

The Game The competition among firms take the shape of a game in which firms compete in supply and demand functions. This means that the players of the game are the firms, and the actions available to each firm i is a vector of supplies for outputs $(S_{k_1 i}, \ldots, S_{k_{d_i^{out}} i})$, demand functions for intermediate inputs $(D_{ij_1}, \ldots, D_{ij_{d_i^{in}}})$, and for labor $\ell_i(\cdot)$ defined over a set $\mathcal{D}_i$ of tuples of input-output prices and stochastic parameter $(p_i, \varepsilon_i) \in \mathcal{D}_i$.

The reason to introduce a stochastic parameter is that this type of modeling has a classical multiplicity problem, as illustrated by Figure 1. The solution, both in the oligopoly and in the market microstructure literature, consists in introducing some source of uncertainty, so that all feasible prices can be realized in equilibrium for some realizations of the uncertainty, and this pins down the full demand or supply schedules rather than just a point on them.

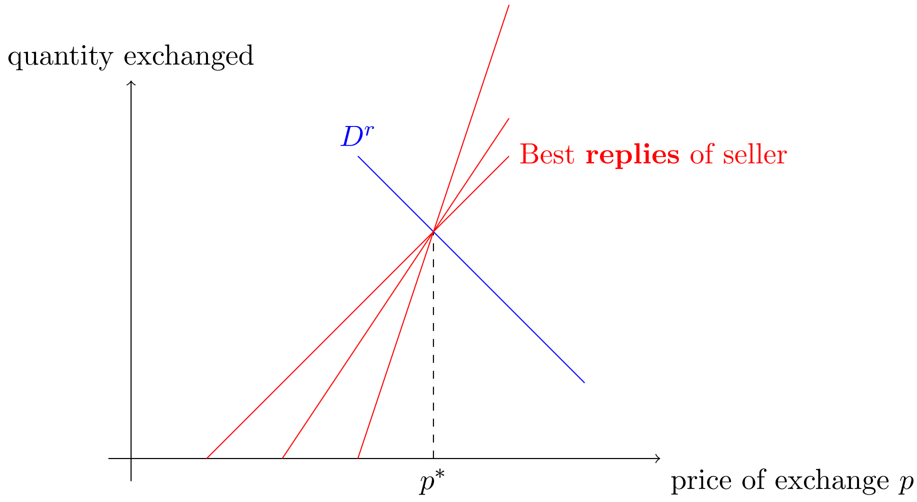
Differently from the Klemperer and Meyer (1989) setting, where a stochastic shock to the exogenous demand function is sufficient to pin down unique best replies, in a supply chain, or more generally in a network economy, more prices have to be determined. This means that the uncertainty in demand alone is not able any more to solve the multiplicity problem. In a network setting, we must add a source of uncertainty in every market, that is one *for every price* to be determined. That will be the role of the productivity shock, shifting the amount of good that a firm is willing to buy from its suppliers and simultaneously the quantity that it is willing to sell.

The feasible supply and demand schedules must:

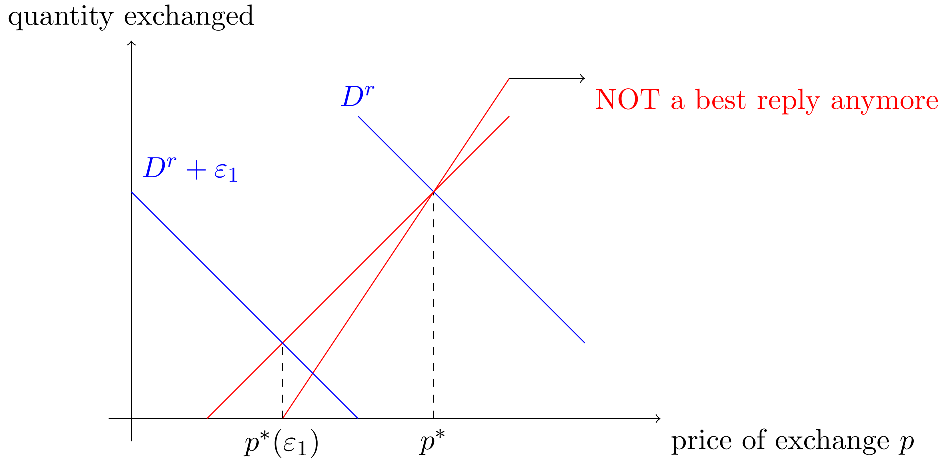
i) be nonnegative;

ii) satisfy the **technology constraint**, that is for any possible (p_i, ε_i):

$$\Phi_i(S_i(p_i, \varepsilon_i), D_i(p_i, \varepsilon_i), l_i(p_i, \varepsilon_i), \varepsilon_i) = 0 \tag{1}$$



(a) If optimal price for seller is p, all red lines represents best replies.



(b) Since the parameter ε_1 is stochastic, the seller will adjust its supply function in order to pin down the optimal price for *any* realization of ε_1, thereby destroying the multiplicity.

Figure 1: Multiplicity problem and solution in Supply Function Equilibrium.

iii) the joint map (S_i, D_i) must be continuously differentiable and have Jacobian derivative with respect to prices $(p_i^{out}, -p_i^{in})$, $J_{i,p_i^{out},-p_i^{in}}$, everywhere positive semidefinite on the support, and has rank at least $d_i - 1$ (the maximum minus 1);[8] note that the differentiation is done with respect to the variables $(p_i^{out}, -p_i^{in})$;[9]

iv) they have a bounded support.

The trades among firms have to satisfie market clearing for any good. These conditions allow us to define the vector of *realized prices* $p^*(\varepsilon)$ through the market clearing equations. The market clearing conditions are:

$$\begin{aligned} \sum_{j,g\to j} D_{jg}(p_j^{out}, p_j^{in}, \varepsilon_j) &= \sum_{k,k\to g} S_{gk}(p_k^{out}, p_k^{in}, \varepsilon_k) \quad g \in M \\ D_{cg}(p_{cg}, \varepsilon_{cg}) &= \sum_{k,k\to g} S_{gk}(p_k^{out}, p_k^{in}, \varepsilon_k) \quad \text{if } g \in C \end{aligned} \tag{2}$$

To show that the regularity conditions indeed imply that the market clearing system can be solved, the crucial step is to show that they translate to regularity conditions on the Jacobian of the function whose zeros define the system above, and then a global form of the implicit function theorem (Krantz and Parks (2012), Theorem 6.2.4) can be applied. This is done in the next Proposition.

Proposition 1. *The market clearing conditions define a function:*

$$p^* : \times_i \mathcal{E}_i \to \mathbb{R}_+^{|E|\times|E|}$$

Note that p^* is also differentiable, but since all the equilibrium analysis and hence the optimizations, will be performed in the linear case, we are not going to use this property in the following.

Now that we built the prices implied by the players's actions, we can define the payoffs. These are the expected profits calculated in the realized prices p^*:

$$\pi_{i\alpha}(S_{i\alpha}, D_{i\alpha}, l_{i\alpha}) = \mathbb{E}\left(\sum_k p_{ki}^* S_{k,i\alpha} - \sum_j p_{ij}^* D_{i\alpha,j} - l_{i\alpha}\right) \tag{3}$$

where to avoid clutter I omitted to write each functional variable.

[8] The Jacobian might not be positive definite because the technology constraint implies, by the chain rule: $\nabla\Phi_{i,S} JS_i + \nabla\Phi_{i,D} JD_i + \nabla\Phi_{i,l} Jl_i = \mathbf{0}$. Depending on how labor enters the technology this might become a linear constraint on the rows of the Jacobian: it is indeed what happens under the parameterization introduced in 2.2, as will be clear in the following.

[9] The matrix $J_{i,p_i^{out},-p_i^{in}}$ is equal to $J_{i,p_i^{out},p_i^{in}}$ but for the fact that all the columns corresponding to input prices have the opposite sign by right multiplication by the matrix

Hence, formally, the game played by firms is: $G = (I, (A_{i\alpha})_{(i,\alpha)\in I}, (\pi_{i\alpha})_{(i,\alpha)\in I}, F)$, where $I = \{(i,\alpha) \mid i = 1, 2,\ \alpha = 1, \ldots n_i\}$ denotes the set of firms, and $A_{i\alpha}$ is the set of profiles of supply and demand functions that satisfie the assumptions above.

Example 1. Standard Supply Function Equilibrium

The model by Klemperer and Meyer (1989) can be seen as a special case of this setting, in which there is only one sector and the network $\mathcal{G}$ is empty, as illustrated in Figure 2. Their setting is a "partial" equilibrium one, in which the consumers do not supply labor to firms but appear only through a demand function $D(\cdot)$, and firms have a cost function for production $C(\cdot)$, that does not explicitly represent payments to anyone. The strategic environment is precisely the same though: if the setting of this section the transformation function is $\Phi_\alpha(q_\alpha, \ell_\alpha) = C^{-1}(q_\alpha) - \ell_\alpha$, and the consumer utility gives rise to demand D, the game G played by firms is precisely the same as in Klemperer and Meyer (1989).

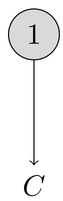


Figure 2: The (degenerate) production network of Example 1: there is only 1 Sector whose firm sell to the consumer.

The welfare of the consumer is $U(C, L)$, where $C(p^*, \varepsilon) = (C_{ci,\alpha}(p_i, \varepsilon_i))_{i,\alpha}$ is the vector of quantities of goods consumed in equilibrium, and $L = \sum_{i,\alpha} l_{i,\alpha}(p_i^*, \varepsilon_i)$ is the total labor used in the economy[10]. The consumers, being atomic, take all prices as given and thus are a non-strategic component of the model, that enter in the game only through their aggregate demand function.

Supply and Demand Function Equilibrium To compute the predictions of the model I just need to specify the role of the stochastic parameters ε. I will use it as a selection device, as made formal by the next definition.

Definition 2.1. A Supply and Demand Function Equilibrium is a profile of prices and quantities of traded goods (p_{ij}, q_{ij}) for all $(ij) \in E$ that realize

[10]It is not necessary to impose a "labor market clearing" condition because it is redundant with the budget constraint of the consumer, consistently with the decision to normalize the wage to 1.

in a Nash Equilibrium of the game G for $F \xrightarrow{\mathcal{D}} 0$:

$$
\begin{aligned}
p_{ij} &= p^*_{ij}(0) \\
q_{ij} &= \sum_\alpha D^*_{i\alpha,j}(p^*_{ij}, 0) \quad \forall (i,j) \in E
\end{aligned}
\tag{4}
$$

So in practice I am using the stochastic variation to “identify” the equilibrium schedules, but when computing the equilibrium predictions I am considering the case in which the shock vanishes.

2.2 Parametric Assumptions

To obtain a tractable solution, I adopt parametric assumptions on the technology. Since firms may produce more than 1 good, I have to express the technology via a *transformation function.* Specifically, assume that the production possibility set of each firm α in sector i be the set of $(q_{k,i\alpha})_k, (q_{i\alpha,j})_j, (l_{i\alpha,kj})_{k,j}$ such that there exists a subdivision $(z_{i\alpha,kj})$ of inputs satisfying $q_{i\alpha j} = \sum_k z_{i\alpha,kj}$, and:

$$
q_{k,i\alpha} = \sum_j \omega_{ij} \min\{\overline{\ell}_{i\alpha,kj}, z_{i\alpha,kj}\} + a\sqrt{\ell k, i\alpha} \quad k = 1, \dots, d^{out}_i \tag{5}
$$

The idea of this functional form is that intermediate inputs $q_{i\alpha,j}$ have to be first allocated to the production of one specific output good: $z_{i\alpha,kj}$ is the amount of input j allocated to the production of the output to be sold to sector k. Moreover, each input needs to be complemented with a specific amount of labor $\overline{\ell}_{i\alpha,kj}$. Labor can be allocated to generic tasks too (we can think to management, organization, anything that is not related to dealing with a specific input), and we denote this amount as $\ell_{k,i\alpha}$. $\overline{\ell}_{i\alpha,kj}$ represents a measure of “effective labor hours”, and is equal to:

$$
\overline{\ell}_{i\alpha,kj} = -\varepsilon_i + \sqrt{\varepsilon_i^2 + 2l_{i\alpha,kj}}
$$

where $l_{i\alpha,kj}$ is the amount labor hired by the firm to deal with input j in the production of output to be sold to k. ε_i is a sector-level labor productivity shock. It changes the marginal product of labor: a large ε_i means that labor is not very productive.

This functional form[11] turns out to be particularly convenient because at the optimum we must have $-\varepsilon_i + \sqrt{\varepsilon_i^2 + 2l_{i\alpha,kj}} = z_{i\alpha,kj}$, so that $l_{i\alpha,kj} =$

[11] A more classical choice, especially in the macro literature, is the one of a production function belonging to the Constant Elasticity of Substitution class. This does not yield tractable expressions here. Notice, however, that the functional form in 5 can be seen as the limit of a *nested* CES:

$$
\sum_j \omega_{ij} \min\{\overline{\ell}_{i\alpha,kj}, z_{i\alpha,kj}\} = \lim_{\substack{\sigma\to\infty,\\ \rho\to 0}} \left(\sum_j \omega_{ij} \left(\left(\overline{\ell}_{i\alpha,kj}(\varepsilon_i)^{\frac{\rho}{(\rho-1)}} + z_{i\alpha,kj}^{\frac{\rho}{\rho-1}} \right)^{\frac{\rho-1}{\rho}} \right)^{\frac{\sigma}{\sigma-1}} \right)^{\frac{\sigma-1}{\sigma}}
$$

$\varepsilon_i z_{i\alpha,kj} + \frac{1}{2} z^2_{i\alpha,kj}$. We can similarly re-define $\sqrt{\ell k, i\alpha} = l^2_{k,i\alpha}$, and the profit function becomes linear-quadratic:

$$\pi_{i\alpha} = \sum_k p_{ki} q_{k,i\alpha} - \sum_j p_{ij} q_{i\alpha,j} - \varepsilon_i \sum z_{i\alpha,kj} - \frac{1}{2} \sum z^2_{i\alpha,kj} - \frac{1}{2} l^2_{k,i\alpha} \quad (6)$$

where the inputs and outputs have to satisfie the technology constraints:

$$q_{i\alpha j} = \sum_k z_{i\alpha,kj} q_{k,i\alpha} = \qquad \sum_j \omega_{ij} z_{i\alpha,kj} + a l_{k,i\alpha}$$

The term $\varepsilon_i \sum z_{i\alpha,kj} + \frac{1}{2} \sum z^2_{i\alpha,kj} + \frac{1}{2} l^2_{k,i\alpha}$ is the cost paid to hire labor. This makes it apparent that ε_i acts reducing the productivity of labor (effective labor hours), and so increasing the amount of labor necessary to achieve the same level of production. This will be crucial in achieving a linear best response.[12]

If a sector uses no intermediate inputs but only labor, the technology is $q_{ki} = \bar{l}_{ki} = -\varepsilon_i + \sqrt{\varepsilon_i^2 + 2 l_{i\alpha,k}}$, so that the profit becomes: $\pi_{i\alpha} = \sum_k p_{ki} q_{k,i\alpha} - \varepsilon_i \sum q_{k,i\alpha} - \frac{1}{2} \sum q^2_{k,i\alpha}$. So we can see that in this case the functional form reduces to a standard technology with quadratic cost function, used for example by Pellegrino (2019), Klemperer and Meyer (1989) and many others.

The analogous assumptions on the utility function of the consumer are that it be quadratic in consumption and (quasi-)linear in disutility of labor L:

$$U((c_i)_i, L) = \sum_i \frac{A_{i,c} + \varepsilon_{i,c}}{B_{c,i}} c_i - \frac{1}{2} \sum_i \frac{1}{B_{c,i}} c_i^2 - L$$

This means that the consumer has demands of the form: $D_{ci} = \max \left\{ A_i - B_{c,i} \frac{p_{ci}}{w}, 0 \right\}$.

Example 2 (Standard Supply and Demand Function equilibrium – parametric). Consider the setting of 1, that is the one sector model. The parametric assumptions in this setting reduce to assuming that the firms have a quadratic cost. Indeed by the same reasoning as above the profit function becomes:

$$\pi_i = pq - \varepsilon_i q - \frac{1}{2} q^2 \quad (7)$$

Graphical intuitions Before delving into the formal details, I will give a graphical illustration of the main mechanisms of the model.

Figure 3 illustrates the mechanics behind the strategic complementarity mechanism. In the left panel, it is shown that the supply function (red line)

[12] It is the analogous in our setting of the assumption of linear/quadratic cost function, common in standard supply function models (Klemperer and Meyer (1989), Delbono and Lambertini (2018)), and the assumption of gaussian random variables and constant absolute risk aversion in the finance setting (Malamud and Rostek (2017), Kyle (1989)).

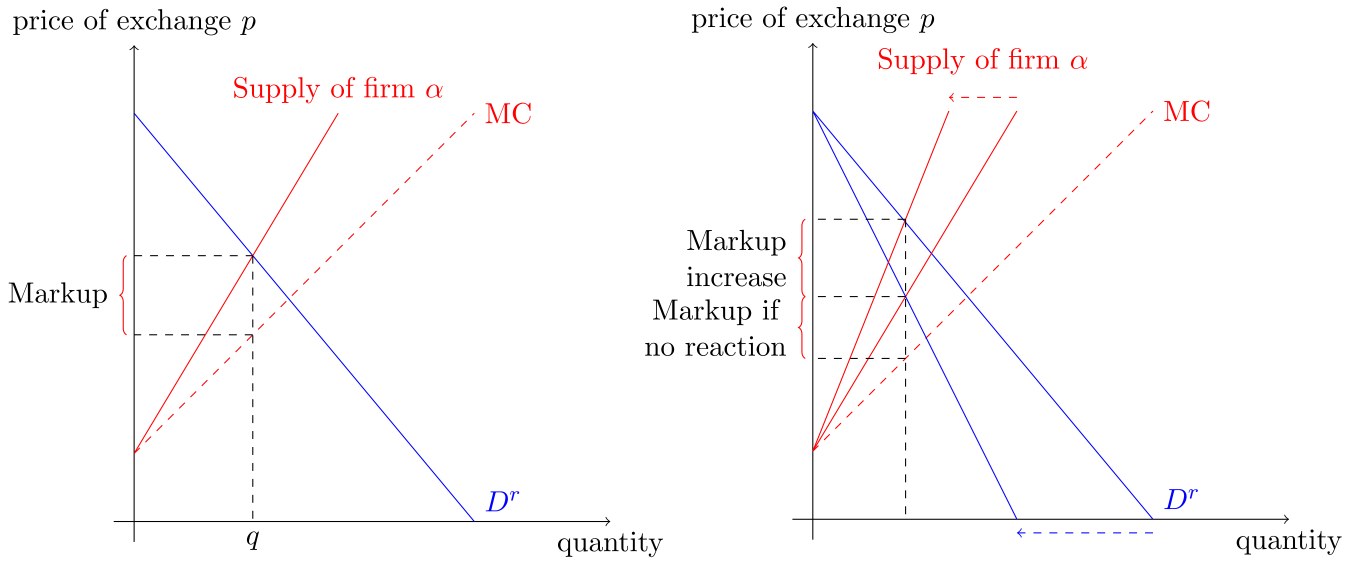


Figure 3: Strategic complementarity among demand and supply.

chosen by a firm as a best reply to the residual demand D^r (the blue line) has to have larger slope than the marginal cost curve, which is the supply function chosen by a firm under perfect competition. The gap between the curves is the (absolute) markup charged by the firm. When the residual supply shifts (right panel), firm α is facing a smaller elasticity, so it wants to increase the markup. To do so it must choose a supply function that is steeper.

3 Solution and Existence

In the following I will focus on S&D equilibria in symmetric linear schedules.

Definition 3.1. A Supply and Demand Function Equilibrium in *symmetric linear schedules* is a profile of functions $\boldsymbol{\sigma} = ((S_{i\alpha})_\alpha, (D_{i\alpha})_\alpha, (l_{i\alpha})_\alpha)_i$ defined on open sets $(\mathcal{P}_{i,\alpha})_{i,\alpha} \times (\mathcal{E}_{i,\alpha})_{i,\alpha}$ such that:

i) $\boldsymbol{\sigma}$ is a Nash Equilibrium of the game G;

ii) (Symmetry) in each sector i firms play the same schedules: $D_{i\alpha} = D_i$, $S_{i\alpha} = S_i$, $l_{i\alpha} = l_i$;

iii) a) (Inactive links) for each i there exists a subset of neighbors $\mathcal{N}_{i,0} \subseteq \mathcal{N}_i$ such that the relative demand or supply function is identically 0; these are called *inactive links*; call the number of active links $d_i^a \leq d_i$, and the prices relative to active links $p_i^a = (p_i^{out,a}, -p_i^{in,a})$;

b) (Linearity) for all i, for all active links $e \in \mathcal{N}_i/\mathcal{N}_{0,i}$ there exist a vector $B_{i,\varepsilon} \in \mathbb{R}^{d_i^a}$ and a matrix $B_i \in \mathbb{R}^{d_i^a \times d_i^a}$ and for all $(p_i, \varepsilon_i) \in \mathcal{E}_i \times \mathcal{P}_i$:

$$\begin{pmatrix} S_i \\ D_i \end{pmatrix} = B_i p_i^a + B_{i,\varepsilon}\varepsilon_i \tag{8}$$

and both $S_i > 0$ and $D_i > 0$ hold.

iv) (feasibility) If $p^*(0)$ is the solution of 2 for $\varepsilon = 0$, then $p_i^*(0) \in \mathcal{P}_i$ for any i.

Note that i implies that B_i is positive definite for all i, because it is the Jacobian of the schedule with respect to $(p_i^{out}, -p_i^{in})$.

This game is in principle very complex to solve, being defined on an infinite-dimensional space. In practice however, things are simpler, because a standard feature of competition in supply schedules, both in the finance and IO flavors, is that the best reply problem can be transformed from an ex-ante optimization over supply functions in an ex-post optimization over input and output prices, as functions of the realizations of the parameter ε_i. In this way the best reply computation is reduced to an optimization over prices as in a monopoly problem. The crucial complication that the input-output dimension adds to e.g. Malamud and Rostek (2017) is the way the residual demand is computed. In an oligopoly without input-output dimension, as in Example 1, the residual demand is the portion of the final demand that is not met by competitors. In our context this remains true, but in computing it, players have to take into account how a variation in quantity supplied affects the balance of trades, hence the prices, in the rest of the network. Let us first define the residual demand in this setting.

Definition 3.2 (Residual demand)**.** Given a profile of linear symmetric schedules $(((S_{i\alpha})_\alpha)_i, (D_{i\alpha})_\alpha)_i$, define the *residual demand*, and the *residual supply* of sector i as the amount of demand and supply remaining once all market clearing conditions but those relative to sector i have been solved. Formally:

$$D_{ik}^r(p_k^{r,i}, p_i, \varepsilon_k, \varepsilon_i) = \underbrace{n_k D_{ki}(p_k^{r,i}, \varepsilon_k)}_{\text{demand from sector } k} - \underbrace{(n_i - 1) S_{ki}(p_i^{out}, p_i^{in}, \varepsilon_i)}_{\text{supply by competitors}}$$
$$S_{ij}^r(p_j^{r,i}, p_i, \varepsilon_j, \varepsilon_i) = \underbrace{n_j S_{ij}(p_j^{r,i}, \varepsilon_j)}_{\text{supply from sector } j} - \underbrace{(n_i - 1) D_i(p_i, \varepsilon_i)}_{\text{demand by competitors}} \quad \forall j, k \in \mathcal{N}_i$$

where $p^{r,i}$ is the *residual price function* of sector i, and is:

i) just the price for all inputs and outputs of i: $p_{ij}^{r,i} = p_{ij}$, $p_{ki}^{r,i} = p_{ki}$;

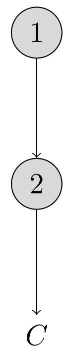


Figure 4: A line production network.

ii) for all other prices, it is the function of p_i and ε defined by the market clearing conditions 2, excluding those relative to the input and output prices of i.

Example 3. (**Line network**)

The easiest setting in which to understand the mechanics of the residual demand is a line network, as illustrated in Figure 4.

What is the residual demand (and supply) in this setting? to understand this, consider a firm in sector 1 that needs to compute its best reply to the schedules chosen by all others. (Details can be found in the Proof of Theorem 1). The demand curve faced by a firm in sector 1 is:

$$\underbrace{n_2 D_2(p_2^*, p_1, \varepsilon_2)}_{\text{Direct demand from sector 2}} - \underbrace{(n_1 - 1) S_1(p_1, \varepsilon_1)}_{\text{Supply of competitors}}$$

for different choices of a supply function $S_{1\alpha}$, different prices p_1 would realize, as functions of the realizations of ε_2. For the best-responding firm, it is equivalent then to simply choose the price p_1 it would prefer for any given ε_1, and then the function $S_{1\alpha}$ can be backed up from these choices. But naturally also p_2^* is determined in equilibrium, and this has to be taken into account when optimizing. In particular, the market clearing conditions for sector 2:

$$n_2 S_{2\alpha}(p_2, p_1, \varepsilon_2) = D(p_2) + \varepsilon_c$$

define implicitly p_2 as a function of p_1 and the shocks. This allows to internalize in the price setting problem of firm 1 the impact that the variation in p_1 is going to have on p_2, for given supply and demand schedules chosen by other players. The same reasoning holds for the supply function. If we assume that all other players are using *linear* supply and demand schedules $S_1(p_1, \varepsilon_1) = B_1(p_1 - \varepsilon_1)$, $D_2(p_2, p_1, \varepsilon_2) = B_2(p_2 - p_1 - \varepsilon_2)$ we get the following expressions for the residual demands:

$$D_1^r = \frac{n_2 B_2}{B_c + n_2 B_2}(A_c + \varepsilon_c - B_c p_1) - (n_1 - 1)B_1(p_1 - \varepsilon_1) \tag{9}$$

$$S_2^r = \frac{n_2 B_2}{B_c + n_2 B_2}(A_c + \varepsilon_c - B_c p_1) - (n_2 - 1)B_2(p_2 - p_1 - \varepsilon_2) \tag{10}$$

$$D_2^r = A_c + \varepsilon_c - B_c p_2 - (n_2 - 1)B_2(p_2 - p_1 - \varepsilon_2) \tag{11}$$

which clarifies how, even if each firms acts "locally" choosing its own input and output prices, actually the problem depends from the parameters of the whole economy.

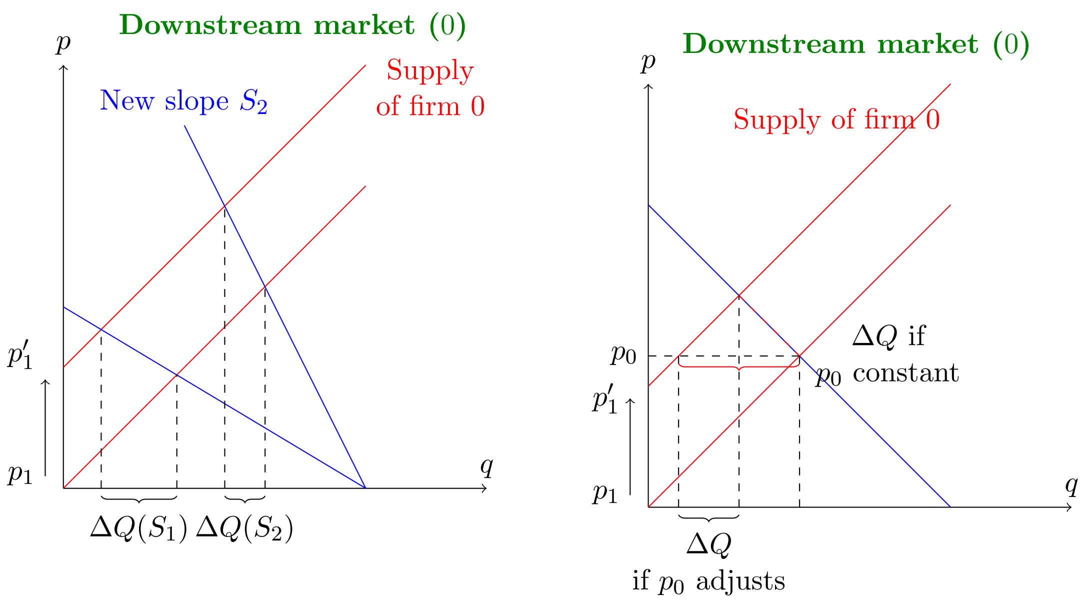


Figure 5: Strategic complementarity across the supply chain.

Figure 5 illustrates how the strategic complementarity extends through the residual demand when firms are indirectly connected through the supply chain. Consider the production network depicted in Figure 4. The slope (and elasticity) of demand that firms in sector 1 face depends on how a variation in price p_1 implies an adjustment in price p_0. A variation in price p_1 implies a shift in the supply curve of firms in sector 0, as the left panel of Figure 5 shows. This implies an upward adjustment of the equilibrium price. The resulting variation in demanded quantity depends on the demand faced by the firms in sector 0, the *downstream market*: the steeper the slope of demand the larger the price adjustment in the downstream market, the smaller the variation in quantity demanded. Hence a large slope of the downstream demand propagates upstream, resulting in a larger slope of demand faced by sector 1. The right panel illustrates that taking into account price adjustment the demand slope perceived is always smaller: this is because all variation is absorbed by quantity, and 0 by the price.

3.1 The input-output matrix

Residual demand and supply are the curves against which each firm will be optimizing when choosing its preferred input and output prices. It is natural therefore that they embed the information about relative market power. The key way through which the structure of the economy (i.e. the network) impacts these functions is via the dependence of the prices p^* on the input and output prices of i. To understand this, consider the market clearing equations.

The market clearing equations 2 define a system:

$$\begin{aligned} S_{il} &= D_{il} \quad \forall i, l \in N,\, i \to l \\ S_{i,c} &= D_{i,c} \quad \forall i \in N,\, i \to c \end{aligned} \tag{12}$$

If all other firms are using symmetric linear schedules with coefficients $(B_i)_i$, then this is a linear system, because all equations are linear in prices. We care about the solution of the system, so the ordering of the equations does not really matter. Let us rewrite the linear supply and demand schedules in a block form as:

$$\begin{pmatrix} S_i \\ D_i \end{pmatrix} = B_i \begin{pmatrix} p_i^{out} \\ -p_i^{in} \end{pmatrix} + \varepsilon_i B_{i,\varepsilon} = \begin{pmatrix} BS_i^{out} & BS_i^{in} \\ BD_i^{out} & BD_i^{in} \end{pmatrix} \begin{pmatrix} p_i^{out} \\ -p_i^{in} \end{pmatrix} + \varepsilon_i B_{i,\varepsilon}$$

(In case the sector employs only labor for production the matrices BD are empty).

So we can rewrite the system 12 as:

$$\begin{aligned} & n_l BS_{l,i\cdot}^{out} p_l^{out} - n_l BD_{l,i\cdot}^{in} p_l^{in} - n_i BD_{i,l\cdot}^{out} p_i^{out} + n_i BD_{i,l\cdot}^{in} p_i^{in} = 0 \quad \forall i, l \in N,\, i \to l \\ & n_i BS_{i,c\cdot}^{out} p_i^{out} - n_i BD_{i,c\cdot}^{in} p_i^{in} + B_{i,c} p_{i,c} = A_{i,c} \quad \forall i \in N,\, i \to c \\ & BS_{l,i\cdot}^{out} p_i^{out} - BD_{l,i\cdot}^{in} p_i^{in} \geq 0 \quad \forall i \in N,\, i \to l, \quad p \geq 0 \end{aligned} \tag{13}$$

To clarify the structure note that the market clearing equation for link $l \to i$ involves all prices of trades in which sectors l and i are involved.

Definition 3.3 (Market clearing coefficient matrix)**.** The *Market clearing coefficient matrix* corresponding to a profile of symmetric linear supply and demand schedules $(S_i, D_i)_i$ is the matrix M of dimension $|L| \times |L|$, where L

is the set of active links according to the profile $(S_i, D_i)_i$, such that for all active links the market clearing system 12 in matrix form is:

$$Mp^a = \boldsymbol{A} + M_\varepsilon \varepsilon \tag{14}$$

$$\tag{15}$$

The vector of constants $\boldsymbol{A}$ is zero but for the entries corresponding to links to the consumer (that have value $\boldsymbol{A}_{ci} = A_{ci}$).

This matrix M is the fundamental source of network information in this setting: it is a matrix indexed on the set of *links* of the network (which correspond to prices and equations in 2), that has a zero whenever two links do not share a node, and p is a vector that stacks all the prices. To have an example, consider the graph in Figure 6 case in which sector 0 has two suppliers: 1 and 2, and 1 itself supplies 2. If the profile of coefficients is $(B_i)_i$, the matrix M (when rows and columns are appropriately ordered) is:

$$\begin{array}{c} \\ (1 \to 0) \\ (2 \to 0) \\ (1 \to 2) \\ (0 \to c) \end{array} \begin{array}{c} \begin{array}{cccc} p_{10} & p_{20} & p_{12} & p_{0c} \end{array} \\ \begin{pmatrix} B_{1,11} + B_{0,22} & B_{0,23} & B_{1,12} & -B_{0,12} \\ B_{0,32} & B_{0,33} + B_{2,11} & -B_{2,12} & -B_{0,13} \\ B_{1,21} & -B_{2,21} & B_{1,22} + B_{2,22} & 0 \\ -B_{0,21} & -B_{0,31} & 0 & B_c + B_{0,11} \end{pmatrix} \end{array}$$

We can see that the only zero is in correspondence of the pair of links $(0, c)$ and $(1, 2)$ which indeed do not share a node.

In network-theoretic language this is the (weighted and signed) adjacency matrix of the *line graph* of the input-output network $\mathcal{G}$. That is the adjacency matrix of the network that has as nodes the link of $\mathcal{G}$ and such that two nodes share a link if and only if the corresponding links in $\mathcal{G}$ have a common sector. Note that this graph is *undirected*, which has the important implication that if all the coefficient matrices B_i are symmetric then also the matrix M is.

To obtain the residual demand, the linear system 12 can be partially solved to yield p^*_{-i} – the vector of all the prices of transactions in which sector i is not directly involved – as a function of p_i:

$$p^{r,i}_{-i} = (M_{-i})^{-1}(-M_{C_i} p_i + \boldsymbol{A}_{-i} + M_\varepsilon \varepsilon)$$

where A_{-i} refers to all the rows of matrix A that do not involve links entering or exiting from node i, and M_{C_i} is the i-th column of M. This can be substituted in the supply and demand functions of suppliers and customers of i to yield the expression in the next proposition.

Proposition 2. *If all firms in all sectors $j \neq i$ are using symmetric linear supply and demand schedules with symmetric positive semidefinite coefficients $(B_j)_j$, generically in the values of $(B_j)_j$ there exist a neighborhood of*

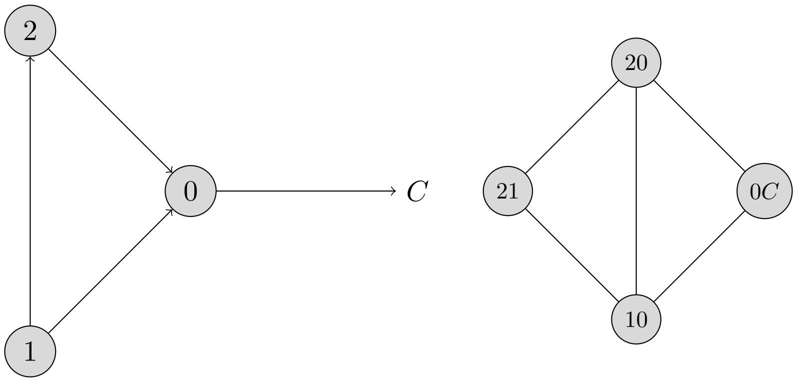


Figure 6: (left) A simple production network: c represents the consumer demand, while the other numbers index the sectors. (Right) The *line graph* of the network nearby.

0 $\mathcal{E}_i$ and a set $\mathcal{P}_i \subset \mathbb{R}^{d_i^a}$ such that the residual supply and demand schedule for active links of sector i is linear and can be written as:

$$\begin{pmatrix} D_i^r \\ S_i^r \end{pmatrix} = -\tilde{\boldsymbol{A}}_i - ((n_i - 1)B_i + \Lambda_i^{-1})p_i^a - \Lambda_{\varepsilon,i}\varepsilon$$

Moreover, Λ_i is symmetric positive definite and equal to the matrix $[M_i^{-1}]_i$, where:

- *M_i is the matrix obtained by M by setting B_i to 0;*
- *if A is a matrix indexed by edges, $[A]_i$ is the submatrix of A relative to all the links that are either entering or exiting i.*

The coefficient Λ_i can be thought as a (sector level) *price impact*[13]: the slope coefficients of the (inverse) supply and demand schedules, describing what effect on prices firms in sector i can have. It is a measure of market power: the larger the price impact, the larger the rents firms in that sector can extract from the market.

Now we can state the theorem. Define the *perfect competition matrix* for sector i as

$$C_i = \begin{pmatrix} \omega_i'\omega_i I^{out} & u_i^{out}\omega_i' \\ \omega_i(u_{out})_i' & d_i^{out} I^{in} \end{pmatrix}$$

Appendix A.3 shows that this is the matrix of coefficients of demands and supplies chosen by a firm that takes prices as given.

Theorem 1. *1. If there are at least 2 firms per sector, generically in the entries of Ω a non-trivial linear and symmetric Supply and Demand Function equilibrium exists;*

[13] Using a financial terminology. It is also the reason for the notation: from Kyle (1989) it is common to denote Λ the price impact of traders.

2. *The equilibrium coefficients $(B_i)_i$ can be written as*

$$B_i = \begin{pmatrix} \tilde{u}_i' \tilde{B}_i \tilde{u}_i & \tilde{u}_i' \tilde{B}_i \\ \tilde{B}_i \tilde{u}_i & \tilde{B}_i \end{pmatrix}$$

for a symmetric positive definite $\tilde{B}_i$ (hence they are positive semi*definite). The equations that characterize them are:*

$$\tilde{B}_i = \left([C_i^{-1}]_{-1} + ((n_i - 1)\tilde{B}_i + \overline{\Lambda}_i) \right)^{-1} \tag{16}$$

where $\overline{\Lambda}_i$ is the constrained price impact*:*

$$\overline{\Lambda}_i = [\Lambda_i^{-1}]_{-1} - \frac{1}{\tilde{u}_i' \Lambda_i^{-1} \tilde{u}_i} [\Lambda_i^{-1} \tilde{u}_i \tilde{u}_i' \Lambda_i^{-1}]_{-1}$$

and the equilibrium prices are all strictly positive: $p > 0$.

The equilibrium coefficients $(B_i)_i$ can be found by iteration of the best reply map, starting:

- *"from above": the perfect competition matrix C_i;*
- *"from below": any sufficiently small (in 2-norm) initial guess.*

The trivial equilibrium in which every supply and demand function are constantly 0, and so no unilateral deviation yields any profit because there would not be trade anyway, is always present[14]. The condition that there are at least two firms in each sector is sufficient but not necessary, indeed in particular cases without the input output dimension it is sufficient that at least three firms participate in any exchange (Malamud and Rostek (2017)).

Part 3) will be important for the numerical solution of the model, as discussed in Section 6.

The *constrained price impact* that appears in equation 16 is the matrix that represents the projection on the space of vectors that satisfy the technology constraint $\sum_k q_{ki} = \sum_j \omega_{ij} q_{ij}$. It is thus the necessary adaptation of the concept to an input-output setting: the technology constraint restricts the degrees of freedom that firms have in impacting the market price.

The expression for the best reply highlights the role of the price impact. If $\Lambda = 0$ then $B_i = C_i$ and the outcome is perfect competition. Moreover, we can see that also if $n_i \to \infty$ the model predicts the perfect competition outcome, as it should.

The proof proceeds in two steps:

[14]This is a feature of the particular technology used, in which labor is a perfect complement to intermediate inputs. In principle if this were not the case a firm producing some final good might find profitable to deviate from the no-trade equilibrium using some labor to sell to the consumers. This would break our assumption on the technology and the linearity of the equilibrium though.

a) I prove that if a profile of matrices $(B_i^*)_i$ satisfies equation 16 on a subnetwork of the original one, and is such that for $\varepsilon = 0$ all implied trades are positive, there exist domains $(\mathcal{E}_i^*, \mathcal{P}_i^*)$ for the linear supply and demand schedules (S_i^*, D_i^*) with coefficients B_i^* such that they are a Nash Equilibrium;

b) I prove that such a profile of matrices exists.

The result can be only stated for generic values of the parameters, and for neighborhoods of $\varepsilon = 0$ because the possibility of corner solutions means that the residual demand in general will only be piecewise linear, and the best reply to piecewise linear strategies in this setting might produce a discontinuous schedule (see Anderson and Hu (2008) for an example). To avoid this technical problems, we consider locally defined schedules. In principle it might be the case that precisely at $\varepsilon = 0$ and $p = p^*(0)$ some best reply has a change in slope: but this happens for non generic values of the parameters.

Step *a*) follows the same principles of Klemperer and Meyer (1989) and Kyle (1989): the infinite dimensional optimization problem over supply and demand functions can be reduced to a finite dimensional one of choosing prices taking the stochastic parameters ε as given. The main difference is that we have the input-output dimension, embodied by the residual demand.

Step b) takes advantage of the fact that the best reply equation for coefficient matrices 16 is increasing in the coefficients of others with respect to the positive semidefinite ordering, hence a converging sequence can be built. This allows to prove also Part 3). Care must be taken because the positive semidefinite ordering does not have the lattice property, and so the techniques of supermodular games cannot be applied directly. Similar techniques are used in Malamud and Rostek (2017).

All proofs are in the Appendix.

Example 4. Networks with no corner solutions

If the network is a tree such that each sector has just one customer sector, as in Figure 7, then it is easy to prove that in equilibrium there is trade on all links. Indeed, by Theorem 1, equilibrium prices are all strictly positive. Then, if i has 0 suppliers, then in equilibrium produces $q_i = B_i p_i > 0$. If sector j has only roots as suppliers, since they all produce strictly positive quantities it follows that $q_j = \sum \omega_{jk} q_k > 0$. Iterating the reasoning we obtain that on all links there is positive trade.

To complete the section, I state two corollaries. The first concerns a partial uniqueness result. Consider sector i, and consider *given* a profile of coefficients of firms in other sectors, that is, consider the sector level price impact Λ_i as given.

Corollary 3.1. If we consider the *sector-level* game played just by firms in sector i, this has a unique linear symmetric equilibrium.

The next corollary shows that in an interior equilibrium we do not need to worry about exit of firms: profits are never negative.

Corollary 3.2. In equilibrium, if quantities are nonnegative, profits can be expressed as:

$$\pi_i = \left((p_i^{out})^*, -(p_i^{in})^*\right) \left(B_i - \frac{1}{2}V_i' C_i V_i\right) \begin{pmatrix} (p_i^{out})^* \\ -(p_i^{in})^* \end{pmatrix}$$

where $V_i = \tilde{C}_i B_i + \frac{1}{k_i}\tilde{u}_i\tilde{u}_i'\Lambda_i^{-1}(I_i - \tilde{C}_i B_i)$

In particular since $B_i - \frac{1}{2}V_i' C_i V_i$ is positive semidefinite, profits are always nonnegative in equilibrium.

3.2 The role of the network

This section describes how the network of input-output relationships affect the equilibrium of the model. The matrix of coefficients of the market clearing system, M, contains the fundamental network information in this setting. The next Proposition shows that the matrix M has a familiar Leontief form.

Proposition 3. *In equilibrium, the matrix M is positive definite, and has positive diagonal and nonpositive off-diagonal elements. In particular, we can write:*

$$M = D - L$$

where D is a positive diagonal matrix, and L a nonnegative matrix with 0 diagonal elements.

Proposition 3 together with the definition of M imply that L is an adjacency matrix of the line graph $\mathcal{L}(\mathcal{G})$ of $\mathcal{G}$, in the sense that it has a nonzero entry only if the links corresponding to row and column share a node. The weights are endogenous, and depend on the equilibrium profile of demand/supply coefficients.

Inverting the matrix M and collecting the diagonal D on both sides we get:

$$M^{-1} = D^{-1/2}(I - D^{1/2}LD^{1/2})^{-1}D^{-1/2}$$

which shows that M^{-1} is, modulo a normalization, has the familiar form of a *Leontief inverse* matrix. It is standard that entries of matrices of this form constitute a measure of the (weighted) number of *undirected* paths connecting the nodes in the network.

Now with the help of Proposition 2, we can understand how the price impact relates to the network. Indeed, according to Proposition 2, to obtain the price impact of say node 2 first we have to eliminate the links of the line graph connecting input and output links of 2. This is equivalent to building the line graph of the *reduced network* $\mathcal{G}_{-2}$, from which we removed the node

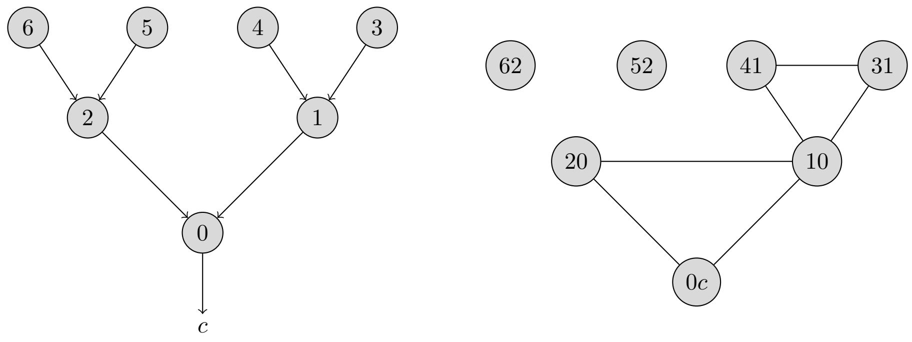


Figure 7: Left: A production network shaped as a regular tree. c represents the consumer demand, while the other numbers index the sectors. Right: the reduced line graph with respect to sector 2.

2. Since this is a tree now we have two separate subnetworks. These are illustrated in Figure 7 (right). Then, by a reasoning similar to Proposition 3 above, the entries of the matrix Λ_2 count the number of weighted paths between input and outputs of 2. But since in the reduced network input and output links are disconnected, the matrix is diagonal, and can be partitioned into:

$$\Lambda_i = \begin{pmatrix} \tilde{D}_i^{-1} & \mathbf{0} \\ \mathbf{0} & \tilde{S}_i^{-1} \end{pmatrix}$$

where $\tilde{D}_i^{-1}$ is the (weighted) number of self loops of the output link in $\mathcal{L}(\mathcal{G})$, and $\tilde{S}_i^{-1}$ is the matrix with on the diagonal the number of self loops of the input links in $\mathcal{L}(\mathcal{G})$.

Figure 8 illustrates the network intuition between the decomposition of Λ. It is very similar to the line network: the more upstream the sector is, the larger the portion of the network in which the "self-loops" have to be calculated. Hence the more elastic the demand it is facing. This is because a larger portion of the network is involved in the determination of the demand, and each price variation will distribute on a larger fraction of firms. The intuition is precisely the reverse for the supply coefficients, represented in Figure 9

Similar reasonings are at work for other networks, with the difference that in general inputs and outputs are not independent in the reduced network. Consider for example the network in Figure 6. What is the price impact of sector 2? In Figure 10 is represented the reduced network. Since now input and output links of sector 2 are connected, this means that Λ_2 is not diagonal anymore.

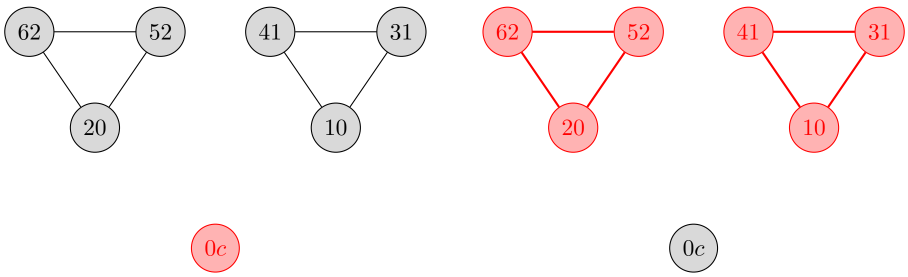


Figure 8: The relevant subnetworks of the line graph $\mathcal{L}(\mathcal{G})$ for the calculation of the price impact of sector 2. Left: output, right: inputs.

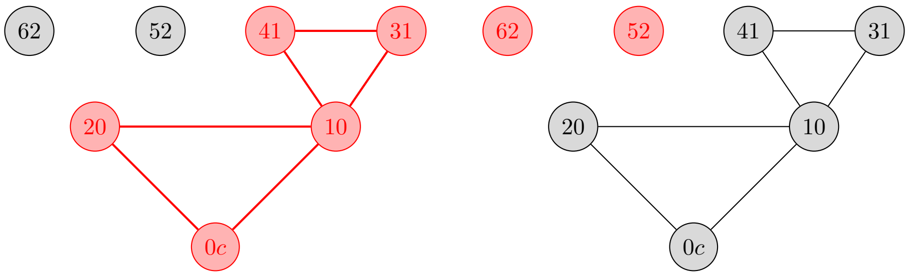


Figure 9: The relevant subnetworks for the calculation of the price impact for sector 0. Left: output, right: inputs.

4 Market power and mergers

In this section I show how the model provides an answer to the question raised in 7, and other examples. I distinguish between *horizontal mergers*, that is mergers between firms that have the same set of neighbors, and *vertical mergers*, that is mergers among firms that are connected through customer-supplier relationships.

As before, a horizontal merger in this setting is simply a decrease in the number of firms, n_i. This is because firms are assumed to be identical and to have no capital, hence the merged firm is ex-ante identical to the non-merged firms, but for the fact that there is one firm less in the market now. The results apply also if firms are not identical, but the new merged firm is identical to one of the merging firms. This describes a situation where, for example, the merging firms have different productivities, but after the merger only the most productive technology is used, so that the after-merger firm is identical to the most productive of the merging firms.

First, I show in an example that revenues are not a sufficient statistics for the impact of mergers in this setting.

Example 5. (**Revenues are not a sufficient statistics for horizontal mergers**)

Consider a tree oriented differently than in Section 7, as in Figure 11,

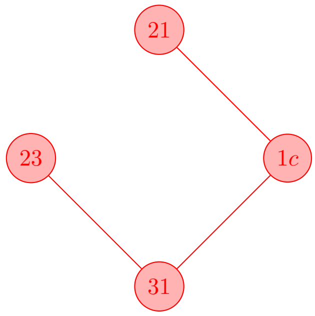


Figure 10: The subnetwork of the line graph in Figure 6 for the calculation of the price impact.

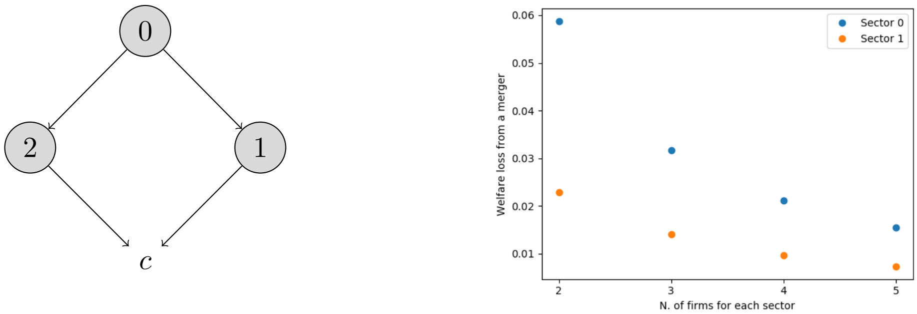


Figure 11: On the Left the network considered in the example, on the Right the welfare loss from a merger for different initial numbers of firms.

consider the case in which the technology is such that $\omega_1 = \omega_2 = \omega_{01} = \omega_{02}$ and all sectors have the same number of firms. In this case parameters are balanced such that *all sectors have the same revenues*. Yet, as in the figure nearby, the welfare loss from mergers is very different in sector 1 and sector 0: it is almost double in sector 0! This shows that a policy maker ignoring the network dimension but focusing only on revenues would choose poorly the sector on which to focus on.

In the context of this paper in any exchange there is *countervailing power*. It is therefore not obvious what a horizontal mergers implies for market power, even in the simple case of no synergies, to which I limit the analysis: the debate goes back to Stigler (1954). As a recent example, Loertscher and Marx (2020) propose a model of bargaining under incomplete information for a bilateral exchange, in which balanced power (for example, an even number of firms on the two sides) increases the efficiency of the exchange. Therefore in their model if there is trade between a very concentrated sector and a very segmented sector, a horizontal merger in the less concentrated sector can be welfare improving because makes power more balanced. This never happens in the Supply and Demand function equilibrium, as the following Theorem illustrates.

Theorem 2. *Assume a horizontal merger does not change the set of active links. Then in the maximal equilibrium it increases all price impacts Λ_i.*

If there is just one consumer good, any merger decreases the quantity consumed.

The mechanism of the theorem is analogous to the one illustrated in Figure 3. The key is that, by the disappearing of a firm, *ceteris paribus*, a merger in any sector is decreasing the slope of all residual supply and demands. This is because direct competitors face less competition, and suppliers/customers, facing less competition, can sustain a higher markup/markdown.

The theorem is formulated via price impacts and prices, because this allows a more general formulation. Example 6 shows how in regular settings, the result can be further precised as: mergers decrease total welfare. In particular, let us focus on the regular tree of Figure 7, and let us assume $\omega_{ij} = \frac{1}{d_i^{in}}$, so that all inputs have the same relative weight in production. Because this choice of technology, this setting allows particularly sharp predictions. This is because, given the symmetry of the problem, all the sectors in the tree will produce the same quantity of output q_i, *no matter the mode of competition*. Hence focusing on this case it is useful can abstract from reallocation and size effects. In the Appendix A.3 I show that in this case under perfect competition profits are identical for all firms.

Example 6. Tree-Total welfare

Consider a tree network such that each sector has only one customer, as in Figure 7, and assume that for each sector all inputs are symmetric, that is $\omega_{ij} = \omega_i$. In this case we can prove that not only the final price increases after a merger, but that also that total welfare decreases. This example is particularly convenient because the symmetric structure implies that total welfare can be expressed as a function of the consumption of final good only:

$$W = \frac{A_c}{B_c} Q_0 - \frac{1}{2B_c} Q_0^2 - \frac{1}{2} \Omega Q_0^2$$

where Ω is a constant that depends on the degree of each node, the number of firms in each sector, and the input-output coefficients. This expression depends only on market clearing and symmetry, so it is true also under perfect competition. In particular, since there is just one consumer good, we know that Q_0 is maximal under perfect competition. But the expression above is increasing if Q_0 is smaller than the maximum, and from this it follows that total welfare also decreases after a merger.

The results for the S&D equilibrium are numerically calculated in Figure 12. It turns out that the equilibrium price impacts are increasing as one moves toward the root of the tree, hence the Corollary above applies in its

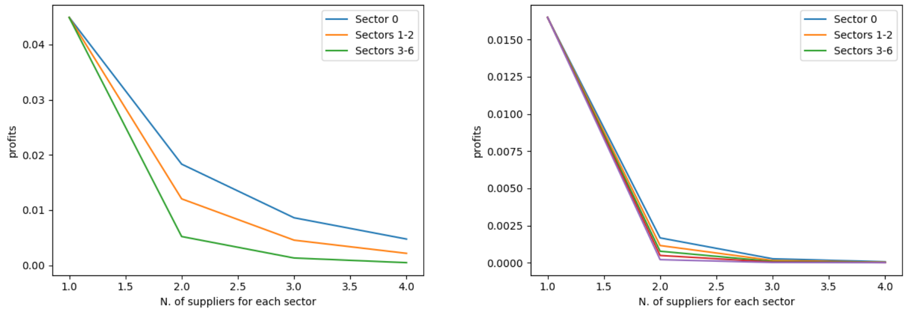


Figure 12: Profits for regular trees of height 2 (Left), and 4 (Right), for different numbers of suppliers. Sector 0 is always making the larger profit (except with 1 supplier, which is the case of the line). The number of firms is set to 2 in each sector.

most useful form. The sector which is the most essential for connecting the whole network is able to extract a larger surplus, and the other are progressively less important the farther upstream one goes.

The importance for the regulator follows the same pattern. Figure 13 shows that the welfare loss from a merger that brings the number of firms from 2 to 1 is larger in sector 0, and smaller the more we move upstream.

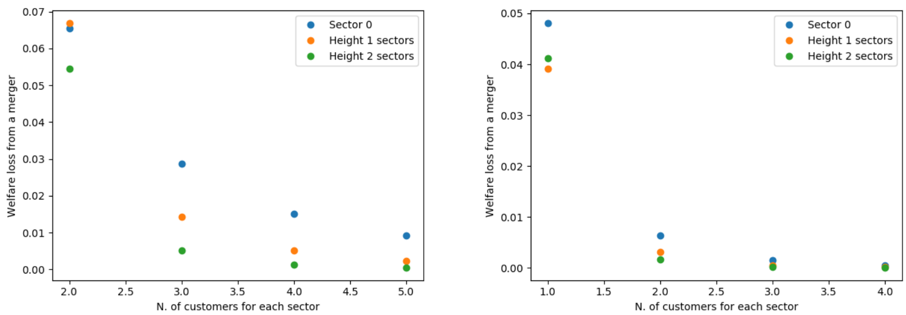


Figure 13: Welfare loss from a merger that brings the number of firms from 2 to 1 in different sectors, for different number of suppliers. Left: tree of height 2, Right: tree of height 4.

4.1 (S&D Equilibrium vs Chain of Oligopolies)

The line network, as in Example 1 is a good setting to gain intuition because sharper results can be obtained. In particular, we can characterize the differences of the sequential competition models with the Supply and Demand Function Equilibrium. Proposition 6 describes how the perceived elasticity of demand tends to decrease as we get closer to the first mover, and why in that type of sequential model upstream firms tend to have larger market power.

What is the analogous of the markup in this bilateral setting? to understand this, let us write the problem of the firm in its general form (as in section 2.2):

$$\max p_i D_i^r(p_i) - p_{i-1} S_i^r - \frac{1}{2} z_i^2 \tag{17}$$

subject to:

$$D_i^r(p_i, p_{i-1}) = z_i \tag{18}$$

$$S_i^r(p_i, p_{i-1}) = z_i \tag{19}$$

this form is naturally redundant in the case of this simple network. Now define μ_i, the *marginal value of inputs* as the Lagrange multiplier relative to the second constraint, and λ_i, the *marginal value of output*, as the Lagrange multiplier relative to the first constraint. Then we can define simultaneously a *markup* and a *markdown*:

$$M_i = p_i - \lambda_i \quad m_i = \mu_i - p_{i-1} \tag{20}$$

which are both zero under perfect competition. The next propostion characterizes their behavior.

Proposition 4. *In a symmetric Supply and Demand Function Equilibrium for the line network, if $n_i = n_j$ for any i, j, then markups are larger the more upstream the sector is, while markdowns are larger the more downstream a sector is.*

This clarifies that the behavior of elasticities in sequential models does not disappear: but here the bilateral nature of the game makes it possible to *both* effects to manifest. How do they balance?

The profit of firms in the symmetric S&D equilibrium can be rewritten as:

$$\pi_i = (M_i + m_i) q_i + \frac{1}{2} q_i^2$$

which makes the intuition transparent: remembering that q_i is constant, the profit in excess of the common component depends on the magnitude of the *sum* of markup and markdown.

Proposition 5. *In a symmetric Supply and Demand Function Equilibrium for the line network, the sector with larger profit is the sector with the smallest number of firms. In particular if $n_i = n_j$ for any i, j, then all sectors have the same profit.*

So, contrary to the sequential competition models, in a S&D equilibrium in a supply chain no one is privileged with respect to others. This follows from the fact that no sector can substitute away from others, they are all essential to produce the consumer good. This allows to shed light on the sequential competition shortcomings: when market power is bilateral one needs to take into account *simultaneously* markup and markdown. When doing so, the paradox disappears and the basic intuition is recovered.

Finally, in 7 I show how vertical mergers present the standard trade-off between foreclosure and decreased double-marginalization, and depending on which effect prevails, they can be welfare-improving or welfare-reducing.

Example 7. Vertical mergers can be welfare improving or not

For a particularly stark example, consider an economy with 2 sectors arranged in a line, with 1 firm in sector 1 and n firms in sector 0. Suppose after a merger between the firm in 1 and a firm in 0 the merged firm does not sell its intermediate good to others but it keeps it all to produce the final output. Then all other firms in 0 cannot produce anymore, and we are left with a monopoly, as shown in Figure 14. The monopoly price in the after-merger setting is:

$$p^M = A\left(B_c + \frac{1}{1+1/B_c}\right)^{-1}$$

where $B^M = \frac{B_c}{1+B_c}$ is the equilibrium coefficient of the supply of the only firm.

In the pre-merger equilibrium instead the final price is:

$$p = \frac{A}{B_c + \frac{nB_0B_1}{nB_0+B_1}} = A\left(B_c + \frac{1}{1+\frac{2}{nB_0}}\right)^{-1}$$

where B_0 and B_1 are as usual the coefficients of the equilibrium supply and demand functions, and the last equality is obtained using the best reply equation for B_1. Hence we get that the price is higher after the merger if and only if $2B_c < nB_0$. The expression shows the trade-off between double marginalization, represented by the factor of 2 that appears because the pre-merger economy is a line with 2 steps, and the extent of foreclosure, represented by nB_0, that measures how much competition is lost after the merger:

$$\underbrace{2}_{\substack{\text{decreased double}\\ \text{marginalization}}} \times B_c < \underbrace{nB_0}_{\text{extent of foreclosure}}$$

If $B_c > 1$, since $B_0 < 1$, for $n = 2$ the merger is welfare-improving. Since the RHS goes to infinity for n sufficiently large it is welfare reducing.

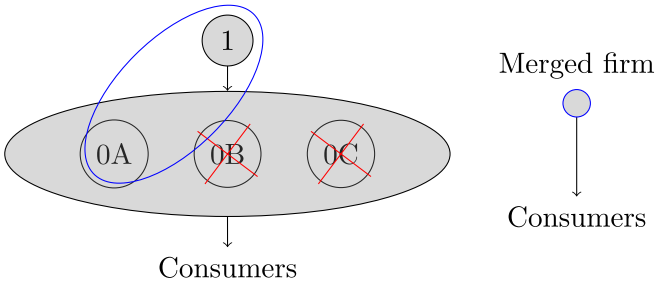


Figure 14: Left: pre-merger economy. The blue circle indicated the merging firms 1 and 0-A. Right: the economy after the merger: B and C are driven out of the market because the merged firm does not sell them the necessary input anymore, and the merged firm becomes a monopolist.

5 Global vs Local Strategic Interactions

In this section I explore the question of how the strategic interactions along the supply chains affect welfare. Taking strategically into account what happens in other sectors has in principle ambiguous effects. We could expect more rational agents to be able to extract more surplus, but on the other hand the effect of an increase in price may be larger because it affects all the chain. Moreover, the firm can change the pattern of markups and markdowns charged, shifting market power towards more vulnerable connections, and this may have non trivial distributional effects. Further, firms trade bilaterally, and their reaction makes in principle the problem hard. Strategic complementarities provide a formidable tool to make welfare comparisons.

First, we need to define the equilibrium with short-sighted firms. The idea is to modify Definition 2.1 and allow firms to neglect the portion of the network they are not directly connected to.

Definition 5.1. A symmetric Local Supply and Demand Function equilibrium is a profile of supply and demand schedules $(S_i, D_i, l_i)_{i\in I}$ such that:

1. the prices and quantities $(p(\varepsilon), q(\varepsilon))$ solve the market clearing conditions when the realization of the shocks is ε;
2. for any firm i, (S_i, D_i, l_i) solves:

$$\max_{(S_{ki})_k,(D_{ij})_j,(z_{i,kj})_{k,j}} \mathbb{E}\left(\sum_k p^*_{ki} S_{ki} - \sum_j p^*_{ij} D_{ij} - \varepsilon_i \sum z_{i\alpha,kj} - \frac{1}{2}\sum_{k,j} z^2_{\alpha,kj}\right) \tag{21}$$

subject to:

$$D_{ki}((p_k^{out}, p_k^{in})^*, \varepsilon) = \sum_k z_{i,kj}((p_i^{out}, p_i^{in})^*, \varepsilon_i), \; \forall i \to k \qquad (22)$$

$$S_{ij}((p_j^{out}, p_j^{in})^*, \varepsilon) = \sum_j \omega_{ij} z_{i,kj}((p_i^{out}, p_i^{in})^*, \varepsilon_i), \; \forall j \to i \qquad (23)$$

$$D_{ki}(p_k^{out}, p_k^{in}, \varepsilon) = S_{ki}((p_i^{out}, p_i^{in})^*, \varepsilon_i), \; \forall i \to k \qquad (24)$$

$$S_{ij}(p_k^{out}, p_k^{in}, \varepsilon) = D_{ij}((p_i^{out}, p_i^{in})^*, \varepsilon_i), \; \forall j \to i \qquad (25)$$

for given actions chosen by the opponents $(S_j, D_j, l_j)_{j \in I\{i\}}$, and *given* prices of other sectors $(p_j)_{j \notin \mathcal{N}_i}$.

The basic difference with Definition 2.1 is that in the firms optimization the *prices of sectors not directly connected with i are taken as given.* Indeed, in the constraints of the optimization there are only the market clearing conditions relative to the links directly connected to i. This is the analogous in this setting of models such as Baqaee (2018), Grassi (2017), Levchenko et al. (2016).

The next theorem explores the welfare implications of this behavioral assumption.

Theorem 3. *In a Local S&D equilibrium, all price impacts are smaller than in the maximal S&D.*

If there is just one consumer good, the price is smaller (the quantity consumed is larger).

Theorem 3 is a qualitative result. In the next Example 8 example I illustrate it quantitatively in the case of a line network. In Example 9, I do a similar comparison for the welfare impact of horizontal mergers.

Example 8. (Welfare effect of Global Strategic Interactions) Consider a line network, as in Figure 4, of length N. Figure 15 depicts total consumer welfare in the global and local versions of the model for different lengths of the line network. As we can see, the gap is increasing in the complexity of the network, and sizable: for a line of length 5 the welfare neglecting intersector strategic effects is 25% larger.

Example 9. Welfare effect of mergers – Local vs Global In this example I show that global strategic interactions are important for the welfare impact of mergers too. I continue to focus on the line network as in the previous example. In this case welfare is simple, because it is just $W = \frac{A_c}{B_c}Q - \frac{1}{2B_c}Q^2 - \frac{N}{2}Q^2$, where N is the number of sectors. Hence the welfare impact of an infinitesimal merger is:

$$\frac{\partial W}{\partial n_i} = \left(\frac{A_c}{B_c} - \left(\frac{1}{B_c} + N \right) Q \right) \frac{\partial Q}{\partial n_i}$$

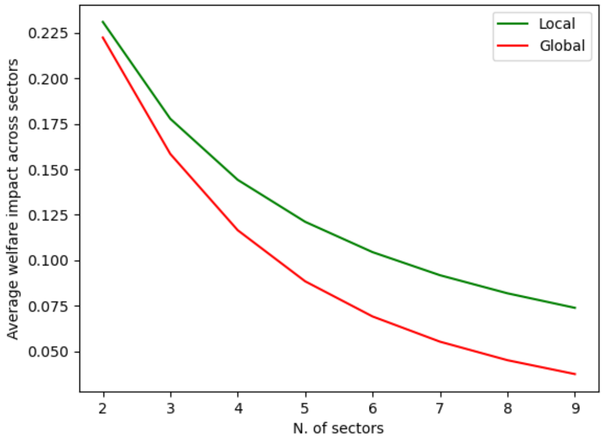


Figure 15: Welfare in the global and local versions of the model for different lengths of the line network. A_c and B_c are fixed to 1, $n = 2$.

The first term represents the fact that the closer Q is to the efficient allocation, the smaller the welfare impact is.

Moreover $Q = \frac{A_c}{B_c}\left(\sum_j \frac{1}{n_j B_j}\right)^{-1}$, so that:

$$\frac{\partial Q}{\partial n_i} = \frac{A_c}{B_c}\left(\sum_j \frac{1}{n_j B_j}\right)^{-2} \sum_j \frac{1}{n_j B_j^2}\frac{\partial B_j}{\partial n_i}$$

To understand the mechanics, let us focus on the simplest case: $n_i = n$ for any i. In this case$B_i = B$ for all i, so that $Q = A_c\left(\frac{1}{B_c} + \frac{N}{nB}\right)^{-1} = A_c \frac{nB}{nB+NB_c}$, and:

$$\frac{\partial Q}{\partial n_i} = Q\left(\frac{nBB_c}{nB+NB_c}\right)\sum_j \frac{1}{nB^2}\frac{\partial B_j}{\partial n_i}$$

$$= Q\frac{B_c}{(nB+NB_c)}\frac{1}{B}\sum_j \frac{\partial B_j}{\partial n_i}$$

and in particular, we see that to compare welfare impacts we need to compare the cumulative effect on the coefficients: $\sum_j \frac{\partial B_j}{\partial n_i}$.

To do this, we differentiate the equilibrium conditions to get a fixed point equation for derivatives. For the Global strategic interaction case:

$$\frac{\partial B_j}{\partial n_i} = (1-B)^2\left(\left(\frac{B_c}{(N-1)B_c+nB}\right)^2\left(n\sum_{k\neq j}\frac{\partial B_k}{\partial n_i} + (1-\delta_{ij})B\right) + \delta_{ij}B + (n-1)\frac{\partial B_j}{\partial n_i}\right)$$

while for the Local case:

$$\frac{\partial B_j}{\partial n_i} = (1-B)^2\left(\left(\frac{1}{2}\right)^2\left(n\frac{\partial B_{j-1}}{\partial n_i}+n\frac{\partial B_{j+1}}{\partial n_i}+(\delta_{i,j+1}+\delta_{i,j-1})B\right)+\delta_{ij}B+(n-1)\frac{\partial B_j}{\partial n_i}\right)$$

Comparing the expressions we see that there are 2 distinct effects at play: one is a "crowding out" effect due to the number of sectors: if N is very large, due to the $(N-1)B_c + nB$ factor in the denominator, in the global version derivatives will tend to be smaller. The other is the strategic interaction effect: in the global case an increase in *any* of the other B coefficient reverberates on any other. The following picture illustrates that the strategic interaction effect can prevail in practice.

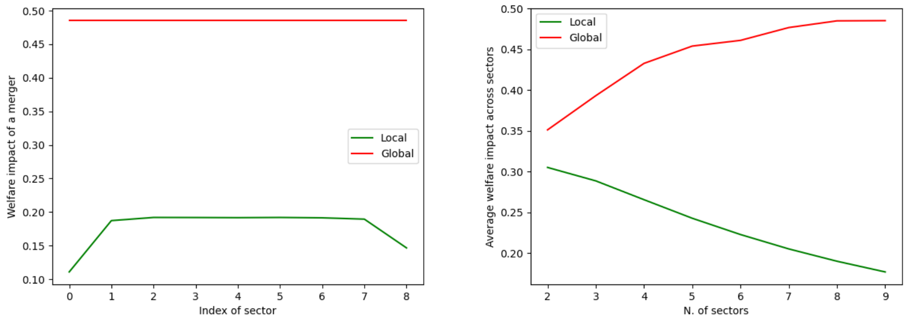


Figure 16: Welfare impact of a merger in the line network. On the right the average (relative) impact for different lengths N, on the left the impact differentiated by sectors for $N = 9$. A_c and B_c are fixed to 1, $n = 2$.

Neglecting the position in the network can lead to evaluating uncorrectly the welfare impact of a merger. In the following example, I show how a vertical merger can be evaluated as welfare improving if firms do not take their position in the network into account, when in the full model is welfare-reducing.

Example 10. Consider the situation illustrated in Example 7. In that context, we can identify a n_* such that the merger is welfare-decreasing if $n > n_*$ (because the foreclosure effect is stronger), and welfare improving if $n < n_*$. Such value is defined implicitly by $B_c = n_* B_0^{global}(n_*)$. Similarly, in the model with local strategic interactions, we can define a similar threshold n_*, defined by $B_c = n^* B_0^{local}(n^*)$. By Theorem 3, $B_0^{\text{local}}(n) > B_0^{\text{local}}(n)$, so that $n_* < n^*$. Hence, it follows that for $n \in (n_*, n^*)$ the merger is welfare decreasing if two sided market power is taken into account, while if only one-sided market power is taken into account is not. it follows that for $n \in (n_*, n^*)$ the merger is welfare decreasing if two sided market power is taken into account, while if only one-sided market power is taken into account is not.

6 Numerical implementation

The solution by iteration of best reply makes the model numerically tractable for medium sized networks. The main bottleneck is the inversion of the market clearing matrix M, which being a matrix links-by-links, tends to be huge, especially if the network is not very sparse. An application of the Matrix inversion lemma (or Woodbury formula, see Horn and Johnson (2012)) allows to invert the full matrix just once, and then at each step update the inverse by just inverting a small matrix, of size equal to the degree of the involved sector. The gain in this process is especially large when the network is sparse because then the matrices to be inverted are small. The algorithm for solving the model numerically is:

1. initialize all the matrices $B_{i,0}$ as C_i;
2. initialize all relative errors of all nodes to some large number, e.g. 1;
3. start from some node $\hat{i}$. Compute the best reply, inverting the matrix M, and save the inverse.
4. choose the node that has the maximum relative error E_i. Compute its best reply. In doing so, update the inverse of the matrix M using the Matrix Inversion Lemma;
5. Repeat 4 until all E_i are smaller than a threshold (I use 0.01).

In Figure 17 I show the computation time to reach the equilibrium for Erdos-Renyi random graphs of 200 nodes, of different densities.

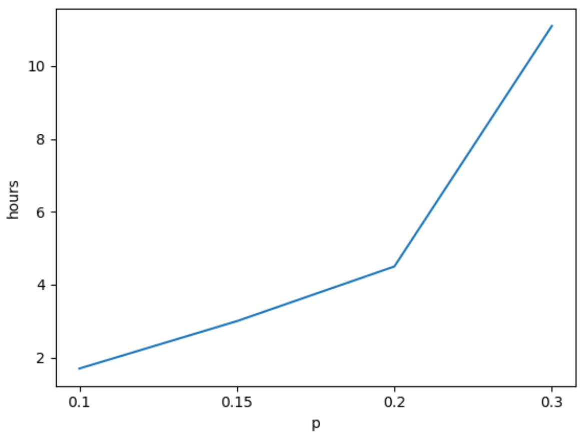


Figure 17: Time of model solution for ER random graphs, 200 nodes, average degree $=200p$. Iteration stopped when maximum percentage error $< 0.1\%$

7 Other benchmarks

In this section I compare the model with some standard benchmarks. The purpose of the section is to show that in most models either firms do not have market power on both inputs and outputs (that is they have *asymmetric market power*), or they do not take into account their position in the supply chain in their decisions (*local strategic considerations*).

7.1 Asymmetric market power

The most-clear cut effect is in a line network as depicted in Figure 4. Below I will consider more general networks.

Assume that goods in each sector are perfect substitutes, and at each stage of the supply chain firms compete à la Cournot, taking as given the input price they face. In our setting this means that firms in sectors 1 and 2 play first, simultaneously, committing to supply a certain quantity. Then firms in sector 0 do the same, taking the price of good 1 and 2 as given. The model can then be solved by backward induction[15].

Call p_0 the inverse demand of the consumer, and assume for simplicity it is concave (this can be sometimes relaxed, as shown below). Assume the technology is linear: $f(q) = Aq$. Capital letters mean sector level quantities, lower case letters are used for firm level quantities.

The markups of firms in sector 0 is equal to the elasticity of the inverse demand, in absolute value. Throughout, I denote elasticities by η:

$$\mu_0 = -\eta_{p_0}$$

What is the markup of upstream sectors? The first order conditions of firms in sector 0 imply that the inverse demand faced by firms in sector 1 is:

$$p_1 = \left(p_0'\left(AQ_1\right)AQ_1 + p_0\left(AQ_1\right)\right)A$$

The markup of firms in sector 1 are then:

$$\begin{aligned}
\mu_1 = -\,\eta_{p_1} &= -\left(\frac{p_0'AQ}{p_0'AQ + p_0}(\eta_{p_0'} + 1) + \frac{p_0}{p_0'AQ + p_0}\eta_{p_0}\right) \\
&= \underbrace{\frac{-p_0'AQ}{p_0'AQ + p_0}}_{>0}(\underbrace{\eta_{p_0'}}_{>0} + 1) + \frac{p_0}{p_0'AQ + p_0}\mu_0 \\
&> \frac{p_0}{p_0'AQ + p_0}\mu_0 > \mu_0
\end{aligned}$$

which puts in evidence that the optimization introduces a force that tends to increase the markup, through the *pass-through*, represented by the term $\frac{p_0}{p_0'f + p_0}$.

[15] This is a version of the simplest setting e.g. in Salinger (1988). A similar model, in prices, is Ordover et al. (1990)

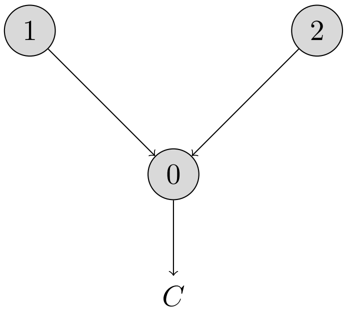


Figure 18: A simple supply chain. Firms in Sectors 1 and 2 sell their output to sector 0 firms, which in turn sell to consumers, denoted by C.

The reasoning can be similarly extended to a chain of any lenght.

More importantly, if there is no compelling physical reason for assuming that firms in sector 1 have precedence over sector 0, an equally reasonable option would be to assume that firms commit to *input* quantities (prices) rather than their output equivalents. An analogous model can then easily be constructed assuming firms in sector 0 decide first, then firms in sector 1. To be more precise, we can compare two different competition structures:

Competition on outputs At $t = 0$ firms in sectors 1 and 2 decide their output quantity; at $t = 1$ firms in sector 0 do the same. Firms in sector 0 face the inverse demand function $p_0(Q_0)$ and all firms take their *input* prices as given.

Competition on inputs At $t = 0$ firms in sector 2 decide their input quantity; at $t = 1$ firms in sector 1 and 2 do the same. Firms in sectors 1 and 2 face the inverse labor supply function $w(L)$ and all firms take their *output* prices as given.

What happens if the network is more general? Consider the case in which 0 has 2 suppliers, as in Figure 18. Its production function would then be $f(q_1, q_2)$. Similarly as above we can derive the inverse demand faced by 1:

$$p_1 = (p_0 f_1 + p_0')f_1$$

The markup of firms in sector 1 are then:

$$\begin{aligned}
\mu_1 =& \eta_{p_1} = -\left(\eta_{f_1} + \frac{p_0' f}{p_0' f + p_0}(\eta_{p_0'}\eta_{f,1} + \eta_{f,1}) + \frac{p_0}{p_0' f + p_0}\eta_{p_0}\eta_{f,1}\right) \\
=& \underbrace{-\eta_{f_1}}_{>0} + \underbrace{\frac{-p_0' f}{p_0' f + p_0}}_{>0}(\underbrace{\eta_{p_0'}}_{>0} + 1)\eta_{f,1} + \frac{p_0}{p_0' f + p_0}\mu_0\eta_{f,1} \\
>& \frac{p_0}{p_0' f + p_0}\mu_0\eta_{f,1} > \mu_0\eta_{f,1}
\end{aligned}$$

which puts in evidence that the optimization introduces a force that tends to increase the markup, through the *pass-through*, represented by the terms η_{f_1} and $\frac{p_0}{p_0' f + p_0}$. The opposing force is the substitution effect, which is driven by $\eta_{f,1}$, the output elasticity of good 1. If this is sufficiently close to 1 with respect to the other terms, we have indeed $\mu_1 > \mu_0$. We can sum up the result in a proposition.

We can sum these results up in a proposition.

Proposition 6. *Consider the supply chain illustrated in Figure 4.1. Assume the consumer has a concave and differentiable demand function, firms have an identical, concave and differentiable constant returns to scale production function f, and there is the same number of firms in each sector. Moreover, assume that at each step of the backward induction the inverse demand remains concave.*

Then:

1. *if the network is a line: under* competition in outputs *firms in sector 1 have larger markup; under* competition in inputs *firms in sector 2 have larger markup.*
2. *if the output elasticity of 0 with respect to input i is close enough to 1, then firms in sector i charge a larger markup than firms in sector 0.*

The conditions are for example satisfied if the technology and utility are quadratic as those used in the main model.

Example 11. (**Markups – Linear-quadratic**)

Assume the firms use the technology introduced in Section 2. Analogously, assume that the consumer demand be $Q_0 = A - Bp_0$. Then the markups are:

$$\mu_0 = \frac{1}{B}\frac{Q_0}{p_0 n_0}$$
$$\mu_1 = \frac{1}{B_1}\frac{Q_1}{p_1 n_1}$$

where

$$B_1 = \left(\frac{1}{n_0} + \frac{\omega_{01}^2}{B}\left(1 + \frac{1}{n_0}\right)\right)^{-1}$$

is the *perceived* slope of demand for upstream firms. This represents the pass-through effect: for $\omega_{01} = 1$ it is always smaller than B_1, but for ω_{01} small (corresponding to a situation where 1 is less important in production), the effect can even be reversed. this happens if:

$$B_1 < B_0 \text{ if and only if } B < n_0(1 - \omega_{01}^2) - \omega_{01}^2$$

Now if $\omega_{01} = \omega_{02}$, that is $q_0 = \frac{1}{2}(q_1 + q_2)$. Under this assumption, if $n_0 = n_1 = n_2$, in equilibrium, $Q_0 = Q_1 = Q_2$, and $p_0 > p_1 = p_2$. As a consequence:

$$B_1 < B_0 \Rightarrow \frac{1}{Bp_0} > \frac{1}{B_1 p_1} \Leftrightarrow \frac{Q_0}{Bp_0 n_0} > \frac{Q_1}{B_1 p_1 n_1} \Rightarrow \mu_1 > \mu_0$$

To make things even more concrete, consider a policy maker that is in charge of evaluating merger proposals. She is constrained in her resources, so she wants to know which are the most important mergers she should focus on. This is a concrete issue: for example in USA it is compulsory to report to the Federal Trade Commission only mergers such that the assets of the firms involved lie above some pre-specified thresholds[16]. On which sectors of the economy should she focus? The next example shows that the choice of competition in inputs vs outputs can radically change things.

Example 12. (**Which is the key sector? – Cobb-Douglas**)

Assume the utility of the consumer is $\frac{Q_0^{1-\alpha}}{1-\alpha} - L$, with $\alpha \in (0,1)$[17], and the technology available in sector 0 is Cobb-Douglas: $f(q_1, q_2) = q_1^{\omega_1} q_2^{\omega_2}$, $\omega_1 + \omega_2 \leq 1$. This technology is a classical choice for production network models, it is used (and generalized) among the others by Grassi (2017), Baqaee and Farhi (2017a).

In this case the output elasticity of input 1 is ω_1. The calculation above applies, but notice that here the inverse demand $p_0 = Q_0^{-\alpha}$ is not concave but convex. The markups are:

$$\mu_0 = \frac{n_0}{n_0 - \alpha} \tag{26}$$

$$\mu_1 = \frac{n_1}{n_1 - \frac{\alpha+1}{2}} \tag{27}$$

So, since $\alpha < \frac{\alpha+1}{2}$, if $n_0 = n_2$ we get $\mu_1 > \mu_0$.

The (log) welfare impact of mergers is a weighted sum of the log variations in markups:

$$\ln C = -\ln p_0 = -\sum_j L_{ij} \ln \mu_j$$

where L is the Leontief inverse matrix of this economy.

[16]Thresholds that recently changed: a change that e.g. Wollmann (2019) argues had a large effect on mergers. This is evidence that the costs are substantial, enough to forgo some regulation to reduce them.

[17]The utility would be well defined and concave for $\alpha \in \mathbb{R}_+$, but for $\alpha > 1$ the relative demand function is inelastic, so cannot be used to model oligopoly because it would yield an infinite price.

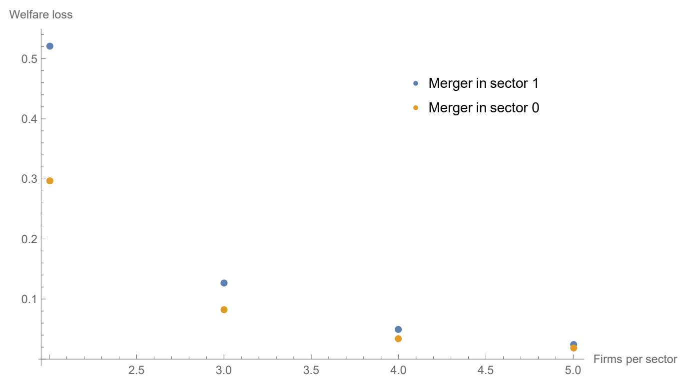


Figure 19: Relative welfare loss from a merger of 2 firms in the network of Figure 18 under the *competition in outputs* with Cobb-Douglas technology. $\alpha = 1/4$.

We can see it analytically for "infinitesimal" mergers, that is small variations in the number n_j treated as a continuum parameter[18]:

$$\frac{\partial \ln C}{\partial n_j} = -L_{ij} \frac{1}{\mu_j} \frac{\partial \ln \mu_j}{\partial n_j}$$

and we see that if α is small enough then mergers in sector 1 are more welfare-damaging than mergers in sector 0. This is because the strategic effect is larger the smaller the α. If it is small enough, it dominates the substitution effect caused by the fact that sector 2 produces a substitute good and its presence diminish the possibility of firms in sector 1 to enjoy rents.

Figure 19 illustrates numerically that this is true also for non-marginal mergers.

The *competition in inputs* does not yield itself to easy analytical characterizations. Figure 20 shows numerically that for small α the sector importance is reversed.

The next example explore a different modeling technique: a bargaining model where the surplus is split according to a parameter δ, and shows that the choice of δ crucially affects relative market power.

Example 13. (**Bargaining**)
In this example I present a simplified variant of the model in Acemoglu and Tahbaz-Salehi (2020), that models the split of surplus between firms in an input-output network using a version of Rubinstein repeated offer game.

[18]Farrell and Shapiro (1990a) interpret infinitesimal mergers as changes in asset structure in an oligopoly. Another interpretation can be that of a small change of concentration, in a context where n_j is a reduced form of a measure of concentration in sector j. Farrell and Shapiro (1990b) also use infinitesimal mergers as an analysis tool

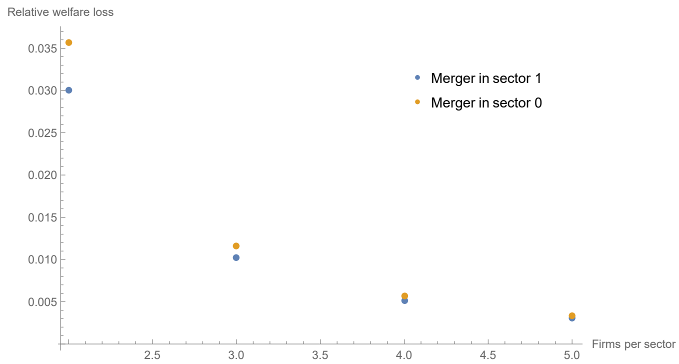


Figure 20: Relative welfare loss from a merger of 2 firms in the network of Figure 18 under the *competition in inputs* with Cobb-Douglas technology. $\alpha = 1/4$.

As a result the allocation and the relative market power depend crucially on the parameters δ_{ij} that capture the relative probability of making the first offer. In this example I neglect the exit dimension that they analyze, to simplify the discussion.

There are 2 firms arranged in a line as in Figure 4. Each firm produces Q from Q inputs, and the consumer provides inelastically 1 units of labor. Assume for simplicity that the relative bargaining power parameter is the same for all input-output relationships, and is δ. The marginal cost of firm 1 is normalized to 0. The solution of the bargaining problem implies that the prices satisfy (by Equation (5) in the reference):

$$\delta(p_0 - p_1) = (1 - \delta)p_1$$
$$A_c - B_c p_0 = 1$$

The immediate calculation shows that the markup of 0 is larger than the markup of 1 if and only if:

$$p_0 - p_1 > p_1 \Longleftrightarrow (1 - 2\delta)p_0 > 0$$

that is, if and only if $\delta < \frac{1}{2}$.

7.2 Local strategic interactions

Most models, mainly in the macroeconomic literature, feature models with *local strategic interactions*. In short, the assumption is that firms internalize the effect of their action on own sector-level variables but not on the other sectors (including suppliers and customers). The purpose of this section is to show that this assumption can greatly affect the welfare impact of oligopoly power.

The modeling technique relies heavily on parametrical assumptions and to the best of my knowledge there is no clear-cut non-parametric definition, so I present it through an example.

Example 14. (**Local strategic interactions – Cobb-Douglas**)
Assume that the technology available to firms is Cobb-Douglas: $f_i(q_{i1}, \dots, q_{in}) = \prod q_{ij}^{\omega_{ij}}$, where q_i is the amount of good i bought. In each sector firms are identical and produce perfect substitutes. The *local strategic interaction* assumption works in this way:

1. firm i chooses the bundle of inputs that minimize costs for any given level of output:
$$q_{ij} = \omega_{ij} \frac{f_i MC_i}{p_j}$$

2. the suppliers of i compete committing to output quantities internalizing the inverse demand:
$$p_j = \omega_{ij} \frac{Q_i MC_i}{Q_{ij}} \tag{28}$$
where $Q_i MC_i$ is *taken as given*.

This procedure is common in production nework models, it is used among the others by Grassi (2017), Baqaee and Farhi (2017a), Levchenko et al. (2016).

To clarify the difference with the sequential approach inspect equation 28: the *perceived* elasticity of demand that the suppliers of i face is 1. This imposes uniformity across the network, in a radically different way with respect to the sequential approach. Indeed, markups are constant and are:

$$\mu_i = \frac{n_i}{n_i - 1} \tag{29}$$

and we can compare with 26 to see that they are always smaller. In the sequential economy, taking strategic considerations into account, the original elasticity of demand shrinks as one moves upstream, while here is artificially fixed to 1.

The (log) welfare impact of mergers is a weighted sum of the log variations in markups:
$$\ln C = -\ln p_0 = -\sum_j L_{ij} \ln \mu_j$$

where L is the Leontief inverse matrix of this economy. It immediately follows that in this economy the welfare loss due to market power is *smaller* than in the sequential economy.

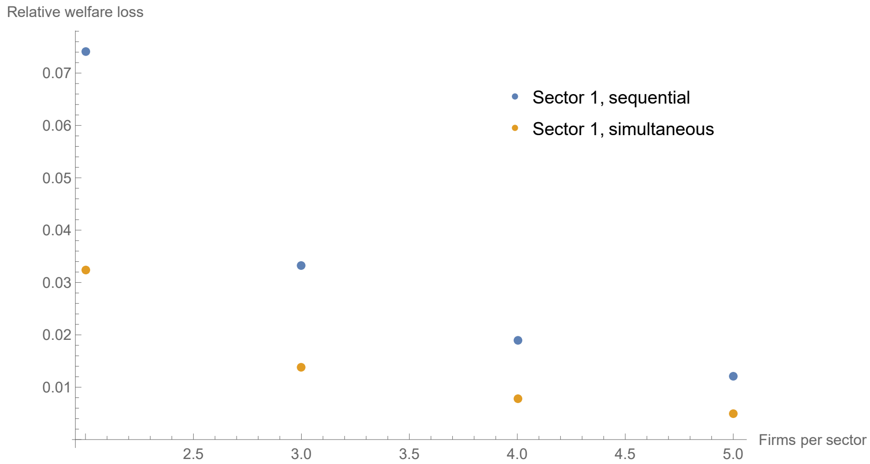


Figure 21: Relative welfare losses from a merger of 2 firms in sector 1 in the network of Figure 18 under the sequential and the local competition with Cobb-Douglas technology. $\alpha = 1/4$.

Not only: also the welfare impact of mergers is larger. As above, we can study it formally for infinitesimal mergers:

$$\frac{\partial \ln C}{\partial n_j} = -L_{ij} \frac{1}{\mu_j} \frac{\partial \ln \mu_j}{\partial n_j}$$

So the information on the welfare impact of mergers is all contained in the markups. $\frac{\partial \ln \mu_j}{\partial n_j}$ is increasing in the elasticity parameter, so also the welfare impact of a marginal merger is larger in the sequential model.

Moreover, the ratio of the increments under the two models can be arbitrarily large as α gets closer to 0, so the difference is sizable.

Figure 21 shows that also a finite (non-marginal) merger has similar properties.

8 Conclusion

I build a model of trade among firms as a game in supply and demand function, which allows to study the problem of how the exogenously given network of firm interactions contributes to determine market power. In the case of a tree network, it is possible to connect the endogenous matrix of price impacts to the intuitive notion of Bonacich centrality. I conjecture that the connection is general. Though Bonacich centrality appears often in input-output economics, in this model I show that not only the size of a firm depends on its position, but also its ability to affect prices. The size of a firm (measured e.g. by revenues) will depend on centrality even under perfet competition, as is well known. Here I am introducing another margin: besides begin large, central firms will have more ability to affect market prices. This is a result that can be of interest in the line of research that explores misallocation and its welfare effects.

The results in Section 6 show that it is actually possible to use this model in networks of a realistic dimension. A full exploration of the insights that can be obtained from real data is an interesting area to develop further. As I have shown through some examples, the model can in principle be used to assess the market impact of mergers as a function of the position in the network, which might be of interest for antitrust authorities.

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Appendix

A Proofs of section 2

A.1 Proof of Proposition 1

The assumption that the consumer demand is zero for very large prices imply that the set of feasible prices is bounded.

Now define the function $F : \mathbb{R}^{|E|\times|E|} \to \mathbb{R}^{|E|\times N}$ (indexed by links) as:

$$F_{ji}(p,\varepsilon) = S_{ji}(p_i,\varepsilon_i) - D_{ji}(p_j,\varepsilon_j) \quad \forall (j,i) \in E \tag{30}$$

$$F_{ci}(p,\varepsilon) = S_{ci}(p_i,\varepsilon_i) - D_{ci}(p_c,\varepsilon_c) \tag{31}$$

so that the market clearing conditions 2 are equivalent to $F(p,\varepsilon) = 0$.

Now I prove that the Jacobian is nonzero, so that the Hadamard implicit function theorem applies.

Call

$$J_i = \begin{pmatrix} JD_i^{out} & JD_i^{in} \\ JS_i^{out} & JS_i^{in} \end{pmatrix}$$

the blocks of the Jacobian matrix, where "in" and "out" refer to the differentiation variables (prices), and S and D to supply and demand. To prove that JF is positive definite, note that row (il) is composed by:

- $JS_{l,ii} + JD_{i,ll}$ in position (il) (diagonal element);
- $JS_{l,ik}$ in position (kl);
- $-JS_{l,ij}$ in position (lj);
- $-JD_{i,lk}$ in position (ki);
- $JD_{i,lj}$ in position (ij).

Consider $x \in \mathbb{R}^{|E|\times|E|}$ and $x'JFx$. Write as usual x_i for $((x_{ki})_{k,i\to k}, (x_{ij})_{j\,j\to i})$. Inspection of the matrix JF reveals that:

$$x'JFx = \sum_m x_m' \hat{J}_m x_m + x_c' J_c x_c$$

where $\hat{J} = \begin{pmatrix} JS_i^{out} & -JS_i^{in} \\ -JD_i^{out} & JD_i^{in} \end{pmatrix}$, that is again positive semidefinite under our assumptions. Now the expression above is nonnegative because a sum of nonnegative terms. The term $x_c' J_c x_c$ is zero only if x_c is zero. Assume the worst case, that all the Jacobians have rank $d_m - 1$. Call $\tilde{u}_m$ the vector that nullifies $\hat{J}_m$. To prove that $x'Jx$ is positive, we have to prove that for any non zero x at least one of the vectors x_m or x_c is non zero and $x_m \neq \hat{u}_m$. If $x_m = \hat{u}_m$ for all m, then the entries of x_c are different from zero and so the expression is positive. If the entries of x_c are all zero, then there is at least 1 of the m such that x_m has a zero entry, and so $x_m' \hat{J}_m x_m > 0$. So we proved that $x'JFx > 0$ if $x \neq 0$, so JF is positive definite.

A.2 Proof of Proposition 2

M is the Jacobian J of Proposition 1 specialized in this linear setting. By the same Proposition, it is invertible and positive definite.

If the supply and demand functions satisfie the conditions of 1, then there exists a set $\mathcal{E}$ such that the price map is defined.

There will be sets $\mathcal{E}_i$ and $\mathcal{P}_i$, such that $\mathcal{E} \subseteq \mathcal{E}_i$ $p_i^*(0) \in \mathcal{P}_i$, such that the partial solution $p_{-i}^*(\varepsilon, p_i)$ is linear. Hence, the residual demand is linear on some set $\mathcal{E}_i \times \mathcal{P}_i$.

Let us calculate it explicitly. we define p_{-i} as the vector of all prices but the prices incident to i. Now we reorder the entries of the matrix M to have in the leading upper left position all the rows that represent equations involving node i, and all the columns relative to prices of input and output of i. Write M_i for the matrix M subject to this reordering. The matrix M can then be partitioned as:

$$M_i = \begin{pmatrix} n_i\tilde{B}_i + B_i^D & M_{R_i} \\ M_{C_i} & M_{-i} \end{pmatrix}$$

where M_{-i} is M from which we cancelled all the rows and columns relative to i, which are M_{R_i} and M_{C_i}, and B_i^D is the matrix with on the diagonal the elements $n_k B_{k,ii}^{out}$ or $n_k B_{k,ii}^{in}$ for all k that are connected to i. Now consider the matrix:

$$\tilde{M}_i = \begin{pmatrix} B_i^D & M'_{C_i} \\ M_{C_i} & M_{-i} \end{pmatrix}$$

The same reasoning proving positive definiteness applied to source nodes shows that at least 2 m are such that $x'_m \hat{B}_m x_m > 0$, so even setting some B_m to zero would not affect invertibility. Hence $\tilde{M}_i$ is still positive definite.

In solving for the objective demand we solve first for p_{-i}:

$$M_{-i}p_{-i} = -M_{C_i}\begin{pmatrix} p_i^{out} \\ p_i^{in} \end{pmatrix} + \boldsymbol{A}_{-i} \Longrightarrow p_{-i} = M_{-i}^{-1}(-M_{C_i}\begin{pmatrix} p_i^{out} \\ p_i^{in} \end{pmatrix} + \boldsymbol{A}_{-i})$$

and then we use it in the expression for objective supplies and demands. The sector level residual demand is, from the market clearing conditions:

$$n_i \begin{pmatrix} S_i \\ D_i \end{pmatrix} = \begin{pmatrix} (n_k D_k)_{k, i\to k} \\ (n_j S_j)_{j, j\to i} \end{pmatrix}$$

Reordering:

$$n_i \begin{pmatrix} -S_i \\ D_i \end{pmatrix} = \begin{pmatrix} -(n_k D_k)_{k, i\to k} \\ (n_j S_j)_{j, j\to i} \end{pmatrix}$$

we can observe that the right hand side corresponds to the market clearing equations 2 for inputs and outputs of i after removing the schedules of sector i. That is, the left hand side corresponds to the first i rows of $\tilde{M}_i p$, that is:

$$[\tilde{M}_i p][\textit{first } i \textit{ rows}] = B_i^D \begin{pmatrix} p_i^{out} \\ p_i^{in} \end{pmatrix} + M'_{C_i} p_{-i} =$$

$$(B_i^D - M'_{C_i} M_{-i}^{-1} M_{C_i}) \begin{pmatrix} p_i^{out} \\ p_i^{in} \end{pmatrix} + M'_{C_i} M_{-i}^{-1} \boldsymbol{A}_{-i}$$

and by block matrix inversion can be seen that $(B_i^D - M_{R_i} M_{-i}^{-1} M_{C_i}) = [(\tilde{M}_i)^{-1}]_i^{-1}$, and moreover is positive definite. To obtain the expression for the residual supply and demand schedule, we have to reorder the signs of the blocks of the coefficient matrix. Define: $\Lambda_i^{-1} = P[(\tilde{M}_i)^{-1}]_i^{-1} P$, where:

$$P = \begin{pmatrix} I & \mathbf{0} \\ \mathbf{0} & -I \end{pmatrix}$$

and we obtain:

$$\begin{pmatrix} D_i^r \\ S_i^r \end{pmatrix} = -\Lambda_i^{-1} \begin{pmatrix} p_i^{out} \\ -p_i^{in} \end{pmatrix} + \tilde{\boldsymbol{A}}_i$$

where $\tilde{\boldsymbol{A}}_i = M'_{C_i} M_{-i}^{-1} \boldsymbol{A}_{-i}$

A.3 Perfect competition benchmark

If a firm takes prices as given will optimize:

$$\max_{q_{k,i\alpha}, q_{i\alpha,j}, z_{i\alpha,kj}} \sum_k p_{ki} q_{k,i\alpha} - \sum_j p_{ij} q_{i\alpha,j} - \frac{1}{2} \sum z_{i\alpha,kj}^2$$

subject to:

$$q_{k,i\alpha} = \sum_j \omega_{ij} z_{i\alpha,kj}, \quad q_{i\alpha,j} = \sum_k z_{i\alpha,kj}$$

The FOC yield:

$$q_{k,i\alpha} = \sum_j \omega_{ij}^2 p_{ki} - \sum_j \omega_{ij} p_{ij} \tag{32}$$

$$q_{i\alpha,j} = \omega_{ij} \sum_k p_{ki} - d_i^{out} p_{ij} \tag{33}$$

Or, in matrix form:

$$q = \begin{pmatrix} \omega'_i \omega_i I_i^{out} & u_{out} \omega'_i \\ \omega_i u'_{out} & d_i^{out} I_i^{in} \end{pmatrix} \begin{pmatrix} p_i^{out} \\ -p_i^{in} \end{pmatrix} = C_i \begin{pmatrix} p_i^{out} \\ -p_i^{in} \end{pmatrix}$$

Moreover, the profit is:

$$\pi_i = \frac{1}{2} \sum_{k,j} (\omega_{ij} p_{ki} - p_{ij})^2 = \frac{1}{2} \sum_{k,j} z_{i,kj}^2$$

and we can see that if firms are all producing the same quantity, as in Section 7, the profits are the same for all.

B Proofs of Section 3

B.1 Proof of Theorem 1

B.1.1 Step a) - A profile of matrices satisfying 16 is a S&D Equilibrium

Rewrite best reply as a finite dimensional optimization Assume all other firms in all other sectors are playing a profile of symmetric linear schedules that for the prices relative to active links have coefficients $(B_j)_j$ which are positive semidefinite. Consider the best reply problem of firm α in sector i. This is:

$$\max_{(S_{ki})_k,(D_{ij})_j,(z_{i,kj})_{k,j}} \mathbb{E}\left(\sum_k p^*_{ki}S_{ki} - \sum_j p^*_{ij}D_{ij} - \varepsilon_i \sum z_{i\alpha,kj} - \frac{1}{2}\sum_{k,j} z^2_{\alpha,kj}\right)$$

subject to the market clearing conditions 2. All the sums run over active links: prices relative to inactive links do not affect the objective function nor the constraints. I already used the fact that at the optimum it must be $l_{i\alpha,kj} = \varepsilon_i z_{i\alpha,kj} + \frac{1}{2}z^2_{i\alpha,kj}$.

Using the residual demand, we can rewrite the optimization as:

$$\max_{(S_{ki})_k,(D_{ij})_j,(z_{i,kj})_{k,j}} \mathbb{E}\left(\sum_k p^*_{ki}S_{ki} - \sum_j p^*_{ij}D_{ij} - \varepsilon_i \sum z_{i,kj} - \frac{1}{2}\sum z^2_{i,kj}\right)$$

subject to:

$$D^r_{ki}((p^{out}_i, p^{in}_i)^*, \varepsilon) = \sum_k z_{i,kj}((p^{out}_i, p^{in}_i)^*, \varepsilon_i),\ \forall i \to k \tag{34}$$

$$S^r_{ij}((p^{out}_i, p^{in}_i)^*, \varepsilon) = \sum_j \omega_{ij} z_{i,kj}((p^{out}_i, p^{in}_i)^*, \varepsilon_i),\ \forall j \to i \tag{35}$$

$$D^r_{ki}(p^{out}_i, p^{in}_i, \varepsilon) = S_{ki}((p^{out}_i, p^{in}_i)^*, \varepsilon_i),\ \forall i \to k \tag{36}$$

$$S^r_{ij}(p^{out}_i, p^{in}_i, \varepsilon) = D_{ij}((p^{out}_i, p^{in}_i)^*, \varepsilon_i),\ \forall j \to i \tag{37}$$

Now assume ε is in the set $\mathcal{E}_i$ where Proposition 2 applies. Then since Λ_i^{-1} is invertible the last two conditions in 34 define uniquely a function for the prices of active links $p^*_i(\varepsilon) : \mathcal{E}_i \to \mathbb{R}^{d_i}$. Then we can rewrite the optimization as:

$$\max_{(S_{ki})_k,(D_{ij})_j,(z_{i,kj})_{k,j},p^*_i} \mathbb{E}\left(\sum_k p^*_{ki}D^r_{ki} - \sum_j p^*_{ij}S^r_{ij} - \varepsilon_i \sum z_{i,kj} - \frac{1}{2}\sum z^2_{i,kj}\right)$$

subject to:

$$D^r_{ki}((p_i^{out}, p_i^{in})^*, \varepsilon) = \sum_k z_{i,kj}((p_i^{out}, p_i^{in})^*, \varepsilon_i),\ \forall i \to k \tag{38}$$

$$S^r_{ij}((p_i^{out}, p_i^{in})^*, \varepsilon) = \sum_j \omega_{ij} z_{i,kj}((p_i^{out}, p_i^{in})^*, \varepsilon_i),\ \forall j \to i \tag{39}$$

$$\tag{40}$$

Now (S, D) do not appear explicitly in the problem any more. For the active links, we can recover them using the information in the pricing function. Indeed, for any x in the range of p_i^*, define:

$$S_{ki}(x, \varepsilon_i) = D^r_{ki}(x, \varepsilon),\ \forall i \to k$$
$$D_{ij}(x, \varepsilon_i) = S^r_{ij}(x, \varepsilon),\ \forall j \to i$$

for some $\varepsilon \in (p_i^*)^{-1}(x)$. By definition of p_i^*, the relation above must be satisfied for all the elements in the counterimage. For all the non-active links, they are both identically zero.

Finally, optimizing with respect to a function of the stochastic variable is equivalent to optimizing ex-post, for any fixed value of ε, as in Klemperer and Meyer (1989). Hence we can write the best reply problem in its final form:

$$\max_{(z_{i,kj})_{k,j}, p_i \in \mathcal{P}_i} \sum_k p_{ki} D^r_{ki}(p_i, \varepsilon) - \sum_j p_{ij} S^r_{ij}(p_i, \varepsilon) - \varepsilon_i \sum z_{i,kj} - \frac{1}{2} \sum z_{i,kj}^2 \tag{41}$$

subject to:

$$D^r_{ki}((p_i, \varepsilon) = \sum_k z_{i,kj},\ \forall i \to k \tag{42}$$

$$S^r_{ij}((p_i, \varepsilon) = \sum_j \omega_{ij} z_{i,kj},\ \forall j \to i \tag{43}$$

$$z_i \geq 0 \tag{44}$$

Optimization The problem 41 in the set $\mathcal{E}_i \times \mathcal{P}_i$ it is a simple concave problem, and can be solved by first order conditions. Now I show that the best reply, defined on $\mathcal{E}_i \times \mathcal{P}_i$, is linear and has as coefficient matrix exactly the B_i^* as defined in 16.

Call λ_{ki} and μ_{ij} the multipliers for input and output constraints respectively, and $J_i = (n_i - 1)B_i + \Lambda_i^{-1}$ the firm level (inverse) price impact. J is the derivative of the supply and demand schedule, and by Proposition 2 it is positive definite.

The Hessian of the problem is a block diagonal matrix with blocks $-(J_i + J_i')$ and minus the identity (with respect to the zs), so the problem is concave.

The FOCs are:

$$p_{hi}: \quad \sum_k \frac{\partial D^r_{ki}}{\partial p_{hi}}(p_{ki} - \lambda_{ki}) - \sum_j \frac{\partial S^o_{ij}}{\partial p_{hi}}(p_{ij} - \mu_{ij}) + D^r_i = 0$$

$$p_{ih}: \quad \sum_k \frac{\partial D^r_{ki}}{\partial p_{ih}}(p_{ki} - \lambda_{ki}) - \sum_j \frac{\partial S^o_{ij}}{\partial p_{ih}}(p_{ij} - \mu_{ij}) - S^r_i = 0$$

$$z_{i,kj}: \quad -\varepsilon_i - z_{i,kj} + \omega_{ij}\lambda_{ki} - \mu_{ij} + t = 0$$

where $t \geq 0$ is the multiplier relative to the constraint $z \geq 0$.

The first set of equations in matrix form reads:

$$J_i \begin{pmatrix} p^{out}_i - \lambda_i \\ -(p^{in}_i - \mu_i) \end{pmatrix} - \begin{pmatrix} D^r_i \\ S^r_i \end{pmatrix} = 0$$

Since this must be true for any prices, and for any price the market clearing conditions must be satisfied, we can rewrite these as:

$$J \begin{pmatrix} p^{out}_i - \lambda_i \\ -(p^{in}_i - \mu_i) \end{pmatrix} = \begin{pmatrix} S_i \\ D_i \end{pmatrix}$$

Now we can use the constraints to get rid first of the z. To do so, sum the derivatives with respect to z to obtain:

$$D^{obj}_{ki} = \sum_j \omega_{ij} z_{i,kj} = \sum_j \omega^2_{ij}\lambda_{ki} - \sum_j \omega_{ij}\mu_{ij} - \sum_j \omega_{ij}\varepsilon_i + \sum_j \omega_{ij} t_{i,kj}$$

$$S^{obj}_{ij} = \sum_k z_{i,kj} = \omega_{ij}\sum_k \lambda_{ki} - d^{out}_i \mu_{ij} - d^{out}_i \varepsilon_i + \sum_k t_{i,kj}$$

Now notice that these equations have a linear dependence, because $\sum_k \sum_j \omega_{ij} z_{i,kj} = \sum_j \omega_{ij} \sum_k z_{i,kj}$. This is not a problem, because $\sum_k D^{obj}_{ki} = \sum_j \omega_{ij} S^{obj}_{ij}$ is indeed a contraint of the problem, but it means that we need to eliminate one equation to solve for the multipliers. Without loss of generality, I eliminate λ_1. If a is a matrix (vector), a_{-1} will denote the elimination of row and column (element) 1. Call $t_{ij} = \sum_k t_{i,kj}$, $t_{ki} = \sum_j t_{i,kj}$. So we can write the system as:

$$\begin{pmatrix} D^{obj}_{i,-1} \\ S^{obj}_i - \lambda_1 \omega_i \end{pmatrix} = \begin{pmatrix} \omega'_i \omega_i I_{-1,out} & u_{-1,out}\omega'_i \\ \omega_i u'_{-1,out} & d^{out}_i I_{in} \end{pmatrix} \begin{pmatrix} \lambda_{i,-1} \\ -\mu_i \end{pmatrix} +$$

$$\begin{pmatrix} t_{ik,-1} \\ t_{ij} \end{pmatrix} - \varepsilon_i \begin{pmatrix} (\omega'_i u^{in}_i) u^{out}_{i,-1} \\ d^{out}_i u^{in}_i \end{pmatrix}$$

Notice that the matrix of the system is $C_{i,-1}$, the $(1,1)-$ minor of the perfect competition matrix.

Solving we get:

$$\begin{pmatrix} \lambda_{i,-1} \\ -\mu_i \end{pmatrix} = C_{i,-1}^{-1} \left[\begin{pmatrix} D_{i,-1}^{obj} \\ S_i^{obj} - \lambda_1 \omega_i \end{pmatrix} - \begin{pmatrix} t_{ik,-1} \\ t_{ij} \end{pmatrix} + \varepsilon_i \begin{pmatrix} (\omega_i' u_i^{in}) u_{i,-1}^{out} \\ d_i^{out} u_i^{in} \end{pmatrix} \right]$$

and for the full vector of multipliers:

$$\begin{pmatrix} \lambda_{i1} \\ \lambda_{i,-1} \\ -\mu_i \end{pmatrix} = \begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{0} & C_{i,-1}^{-1} \end{pmatrix} \left[\begin{pmatrix} 0 \\ D_{i,-1}^{obj} \\ S_i^{obj} \end{pmatrix} - \begin{pmatrix} 0 \\ t_{ik,-1} \\ t_{ij} \end{pmatrix} + \varepsilon_i \begin{pmatrix} 0 \\ (\omega_i' u_i^{in}) u_{i,-1}^{out} \\ d_i^{out} u_i^{in} \end{pmatrix} + \lambda_{1i} \begin{pmatrix} 1 \\ \mathbf{0} \\ -\omega_i \end{pmatrix} \right]$$

Now using the constraint $(u_{out}', -\omega_i') \begin{pmatrix} S_i \\ D_i \end{pmatrix} = 0$ we can rewrite:

$$(u_{out}', -\omega_i') J_i \begin{pmatrix} p_i^{out} \\ -p_i^{in} \end{pmatrix} = (u_{out}', -\omega_i') J_i \begin{pmatrix} \lambda_i \\ -\mu_i \end{pmatrix}$$

and substituting the multipliers we get:

$$\tilde{u}_i' J_i \begin{pmatrix} p_i^{out} \\ -p_i^{in} \end{pmatrix} = \tilde{u}_i' J_i \begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{0} & C_{i,-1}^{-1} \end{pmatrix} \left[\begin{pmatrix} 0 \\ D_{i,-1}^{obj} \\ S_i^{obj} \end{pmatrix} - t_{i,-1} + \varepsilon_i \begin{pmatrix} 0 \\ (\omega_i' u_i^{in}) u_{i,-1}^{out} \\ d_i^{out} u_i^{in} \end{pmatrix} + \lambda_{1i} \begin{pmatrix} 1 \\ \mathbf{0} \\ -\omega_i \end{pmatrix} \right]$$

$$-\lambda_{1i} \tilde{u}_i' J_i \begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{0} & C_{i,-1}^{-1} \end{pmatrix} \begin{pmatrix} 1 \\ \mathbf{0} \\ -\omega_i \end{pmatrix} =$$

$$\tilde{u}_i' J_i \left[\begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{0} & C_{i,-1}^{-1} \end{pmatrix} \left(\begin{pmatrix} 0 \\ D_{i,-1}^{obj} \\ S_i^{obj} \end{pmatrix} - t_{i,-1} + \varepsilon_i \begin{pmatrix} 0 \\ (\omega_i' u_i^{in}) u_{i,-1}^{out} \\ d_i^{out} u_i^{in} \end{pmatrix} \right) - \begin{pmatrix} p_i^{out} \\ -p_i^{in} \end{pmatrix} \right]$$

Now the inverse of $C_{i,-1}$ can be calculated through block inversion and Sherman-Morrison formula, and is:

$$C_{i,-1}^{-1} = \begin{pmatrix} \frac{1}{\omega_i' \omega_i} \left(I_{-1,i}^{out} + u_{-1,out} u_{-1,out}' \right) & -\frac{1}{\omega_i' \omega_i} u_{-1,out} \omega_i' \\ -\frac{1}{\omega_i' \omega_i} \omega_i u_{-1,out}' & \frac{1}{d_i^{out}} \left(I_{in} + \frac{d_i^{out} - 1}{\omega_i' \omega_i} \omega_i \omega_i' \right) \end{pmatrix}$$

from which we get:

$$\begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{0} & C_{i,-1}^{-1} \end{pmatrix} \begin{pmatrix} 1 \\ \mathbf{0} \\ -\omega_i \end{pmatrix} = \begin{pmatrix} u_{out} \\ -\omega_i \end{pmatrix}$$

and

$$\begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{0} & C_{i,-1}^{-1} \end{pmatrix} \begin{pmatrix} 0 \\ (\omega_i' u_i^{in}) u_{i,-1}^{out} \\ d_i^{out} u_i^{in} \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ u_i^{in} \end{pmatrix}$$

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So that the coefficient of λ_{1i} is $k_i = \tilde{u}_i' J_i \tilde{u}_i > 0$. Substituting this into the expression for the multiplier:

$$\begin{pmatrix} \lambda_{i1} \\ \lambda_{i,-1} \\ \mu_i \end{pmatrix} = \begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{0} & C_{i,-1}^{-1} \end{pmatrix} \times \left[\left(\left(\begin{pmatrix} 0 \\ D_{i,-1}^{obj} \\ S_i^{obj} \end{pmatrix} - t_{i,-1} + \varepsilon_i \begin{pmatrix} 0 \\ (\omega_i' u_i^{in}) u_{i,-1}^{out} \\ d_i^{out} u_i^{in} \end{pmatrix} \right) \right) \right.$$

$$\left. -\frac{1}{k} \begin{pmatrix} 1 \\ \mathbf{0} \\ -\omega_i \end{pmatrix} \tilde{u}_i' J_i \left[\begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{0} & C_{i,-1}^{-1} \end{pmatrix} \left(\left(\begin{pmatrix} 0 \\ D_{i,-1}^{obj} \\ S_i^{obj} \end{pmatrix} - t_{i,-1} + \varepsilon_i \begin{pmatrix} 0 \\ (\omega_i' u_i^{in}) u_{i,-1}^{out} \\ d_i^{out} u_i^{in} \end{pmatrix} \right) \right) - \begin{pmatrix} p_i^{out} \\ -p_i^{in} \end{pmatrix} \right] \right]$$

so the expression above becomes:

$$\begin{pmatrix} \lambda_{i1} \\ \lambda_{i,-1} \\ \mu_i \end{pmatrix} = \left(I_i - \frac{1}{k_i} \tilde{u}_i \tilde{u}_i' J_i \right) \begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{0} & C_{i,-1}^{-1} \end{pmatrix} \left(\left(\begin{pmatrix} 0 \\ D_{i,-1}^{obj} \\ S_i^{obj} \end{pmatrix} - t_{i,-1} \right) + \varepsilon_i \begin{pmatrix} \mathbf{0} \\ u_i^{in} \end{pmatrix} \right.$$

$$+ \frac{1}{k_i} \tilde{u}_i \tilde{u}_i' J_i \begin{pmatrix} p_i^{out} \\ -p_i^{in} \end{pmatrix}$$

So we can finally substitute and get the demand function (after using market clearing to turn objective into supply and demand). So:

$$\begin{pmatrix} S_i \\ D_i \end{pmatrix} = J_i \begin{pmatrix} p_i^{out} \\ -p_i^{in} \end{pmatrix} -$$

$$J_i \left(\left(I_i - \frac{1}{k_i} \tilde{u}_i \tilde{u}_i' J_i \right) \left(\begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{0} & C_{i,-1}^{-1} \end{pmatrix} \left(\left(\begin{pmatrix} 0 \\ D_{i,-1}^{obj} \\ S_i^{obj} \end{pmatrix} - t_{i,-1} \right) + \varepsilon_i \begin{pmatrix} \mathbf{0} \\ u_i^{in} \end{pmatrix} \right) \right) + \frac{1}{k_i} \tilde{u}_i \tilde{u}_i' J_i \begin{pmatrix} p_i^{out} \\ -p_i^{in} \end{pmatrix} \right)$$

Now to re-express everything in terms of supply and demand functions note that:

$$\begin{pmatrix} 0 \\ D_{i,-1}^{obj} \\ S_i^{obj} \end{pmatrix} = \begin{pmatrix} 1 & u_{-1,out}' & -\omega_i' \\ \mathbf{0} & I_{-1,out} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & I_{in} \end{pmatrix} \begin{pmatrix} D_i^{obj} \\ S_i^{obj} \end{pmatrix}$$

call:

$$\tilde{C}_i = \begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{0} & C_{i,-1}^{-1} \end{pmatrix} \begin{pmatrix} 1 & u_{-1,out}' & -\omega_i' \\ \mathbf{0} & I_{-1,out} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & I_{in} \end{pmatrix} = \begin{pmatrix} 1 & \tilde{u}_{i,-1}' \\ \mathbf{0} & C_{i,-1}^{-1} \end{pmatrix}$$

[19] This can be seen by the explicit calculation of:

$$\begin{pmatrix} 1 \\ \mathbf{0} \\ -\omega_i \end{pmatrix} = \begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{0} & C_{i,-1} \end{pmatrix} \begin{pmatrix} u_{out} \\ -\omega_i \end{pmatrix}$$

and eventually we get:

$$\begin{pmatrix} S_i \\ D_i \end{pmatrix} = \left(I_i + \left(J_i - \frac{1}{k_i} J_i \tilde{u}_i \tilde{u}_i' J_i\right) \tilde{C}_i\right)^{-1} \left(J_i - \frac{1}{k_i} J_i \tilde{u}_i \tilde{u}_i' J_i\right) \left(\begin{pmatrix} p_i^{out} \\ -p_i^{in} \end{pmatrix} - \varepsilon_i \begin{pmatrix} \mathbf{0} \\ u_i^{in} \end{pmatrix} + t_i\right)$$

where $t_i = \begin{pmatrix} 0 \\ C_{i,-1}^{-1} t_{i,-1} \end{pmatrix}$.

To obtain the expression in the text of the Theorem, notice that $\left(J_i - \frac{1}{k_i} J_i \tilde{u}_i \tilde{u}_i' J_i\right)$ is not invertible because not all equations are independent. Let us solve for the last $d^{in} + d^{out} - 1$ equations:

$$\begin{pmatrix} S_{i,-1} \\ D_i \end{pmatrix} = (I + J_{-R_1} \left(I - \frac{1}{k_i} \tilde{u}\tilde{u}' J\right)_{-C_1} \tilde{C}_i)^{-1} \times$$

$$J_{-R_1} \left(I - \frac{1}{k} \tilde{u}\tilde{u}' J\right) \left(\begin{pmatrix} p_i^{out} \\ -p_i^{in} \end{pmatrix} + t_i - \varepsilon_i \begin{pmatrix} \mathbf{0} \\ u_i^{in} \end{pmatrix}\right)$$

Now $J - \frac{1}{k} J\tilde{u}\tilde{u}' J$ is positive semidefinite. To see this, note that $x'(J - \frac{1}{k} J\tilde{u}\tilde{u}' J)x \geq 0$ if and only if $(\tilde{u}' J\tilde{u})(x' Jx) \geq (\tilde{u}' Jx)(x' J\tilde{u})$, which follows from Cauchy Schwartz inequality[20] Moreover canceling the first row and column yields a positive definite matrix, because in that case $1 - \frac{1}{\tilde{u}J\tilde{u}'} \tilde{u}_{-1} J_{-1} \tilde{u}_{-1}' > 0$. Inverting we get:

$$\begin{pmatrix} S_{i,-1} \\ D_i \end{pmatrix} = \left(\left(J - \frac{1}{k_i} J\tilde{u}\tilde{u}' J\right)_{-1}^{-1} + C_{i,-1}^{-1}\right)^{-1} \times$$

$$\left(\left(J - \frac{1}{C} J\tilde{u}\tilde{u}' J\right)_{-1}^{-1} J_{-R_1} \left(I - \frac{1}{k} uu' J\right)_{C_1}, I_{-1}\right) \left(\begin{pmatrix} p_i^{out} \\ -p_i^{in} \end{pmatrix} + t_i - \varepsilon_i \begin{pmatrix} \mathbf{0} \\ u_i^{in} \end{pmatrix}\right)$$

$$= \left(\left(J - \frac{1}{k} J\tilde{u}\tilde{u}' J\right)_{-1}^{-1} + C_{i,-1}^{-1}\right)^{-1} \left(J - \frac{1}{C} J\tilde{u}\tilde{u}' J\right)_{-1}^{-1} J_{-R_1} \left(I - \frac{1}{k} \tilde{u}\tilde{u}' J\right)_{C_1} p_{i,1}^{out} +$$

$$\left(\left(J - \frac{1}{C} J\tilde{u}\tilde{u}' J\right)_{-1}^{-1} + C_{i,-1}^{-1}\right)^{-1} \left(\begin{pmatrix} p_{i,-1}^{out} \\ -p_i^{in} \end{pmatrix} + t_i - \varepsilon_i \begin{pmatrix} \mathbf{0}_{-1} \\ u_i^{in} \end{pmatrix}\right)$$

Now the null space of $J - \frac{1}{\tilde{u}' J\tilde{u}} J\tilde{u}\tilde{u}' J$ is parallel to $\tilde{u}$, since $(J - \frac{1}{\tilde{u}' J\tilde{u}} J\tilde{u}\tilde{u}' J)u = J\tilde{u} - J\tilde{u} = 0$. Hence we have that

$$J_{-R_1} \left(I - \frac{1}{k} \tilde{u}\tilde{u}' J\right)_{C_1} = -\sum_j J_{-R_1} \left(I - \frac{1}{k} \tilde{u}\tilde{u}' J\right)_{C_j} = -\left(J - \frac{1}{k} J\tilde{u}\tilde{u}' J\right)_{-1} \tilde{u}_{-1}$$

[20] Which holds even if matrices are not symmetric. To see this:

$$\left(x - \frac{x' Ju}{u' Ju} u\right)' J \left(x - \frac{x' Ju}{u' Ju} u\right) = x' Jx + \left(\frac{x' Ju}{u' Ju}\right)^2 u' Ju - \frac{x' Ju}{u' Ju} (x' Ju + u' Jx)$$

$$= x' Jx - \frac{x' Ju}{u' Au} u' Ju$$

which is nonnegative if J is positive semidefinite.

hence we get the final expression for supplies and demands:

$$\begin{pmatrix} S_{i,-1} \\ D_i \end{pmatrix} = \left(\left(J_i - \frac{1}{k_i} J_i \tilde{u}_i \tilde{u}_i' J_i \right)_{-1}^{-1} + C_{i,-1}^{-1} \right)^{-1} \left(-p_i^{out} \tilde{u}_{-1} + \begin{pmatrix} p_{i,-1}^{out} \\ -p_i^{in} \end{pmatrix} + t_i - \varepsilon_i \begin{pmatrix} \mathbf{0} \\ u_i^{in} \end{pmatrix} \right)$$

Finally, note that in equilibrium

$$\tilde{B}_i \tilde{u}_i = (I_i + (J_i - \frac{1}{k_i} J_i \tilde{u}_i \tilde{u}_i' J_i) \tilde{C}_i)^{-1} \left(J_i - \frac{1}{k_i} J_i \tilde{u}_i \tilde{u}_i' J_i \right) \tilde{u}_i = 0$$

hence $J_i - \frac{1}{k_i} J_i \tilde{u}_i \tilde{u}_i' J_i = (n_i - 1) B_i + \overline{\Lambda}_i$

B.1.2 Step b) - A profile of matrices satisfying 16 exists

Now I have to show that a non-trivial profile of matrices satisfying 16 exists, exhibiting sequences that converge to it.

Increasing best reply Assume B and B' are two profiles of schedules such that $B_i' > B_i$ in the Loewner (positive semidefinite) order for any i. The best reply is a function of B_i and $\overline{\Lambda}_i$ through a double inversion, so is increasing in both. Hence, to prove that the best reply is increasing in the positive semidefinite ordering we have to prove that $\overline{\Lambda}_i$ is increasing in the profile B.

In the notation of Proposition 2, $\hat{B}_i$ is increasing in the Loewner order. Indeed:

$$\hat{B}_i' = \begin{pmatrix} CS_i^{out} & CS_i^{in} \\ CD_i^{out} & CD_i^{in} \end{pmatrix} > \begin{pmatrix} BS_i^{out} & BS_i^{in} \\ BD_i^{out} & BD_i^{in} \end{pmatrix} = \hat{B}_i$$

if and only if

$$\begin{pmatrix} CS_i^{out} - BS_i^{out} & CS_i^{in} - BS_i^{in} \\ CD_i^{out} - BD_i^{out} & CD_i^{in} - BD_i^{in} \end{pmatrix} > 0$$

which is true if and only if

$$\begin{pmatrix} CS_i^{out} - BS_i^{out} & -(CS_i^{in} - BS_i^{in}) \\ -(CD_i^{out} - BD_i^{out}) & CD_i^{in} - BD_i^{in} \end{pmatrix} > 0$$

Since $\hat{B}_i$ is increasing, also the market clearing matrix M is increasing, because remember from 2 that $x'Mx = \sum_m x_m' \hat{B}_m x_m$.

Now $\hat{M}^{-1}$ is decreasing. Canceling rows and columns does not change the Loewner ordering, and so $\Lambda_i^{-1} = (\hat{M}^{-1})_i^{-1}$ is increasing.

Finally, I prove that $\overline{\Lambda}_i = \Lambda_i^{-1} - \frac{1}{k_i} \Lambda_i^{-1} \tilde{u}_i \tilde{u}_i' \Lambda_i^{-1}$ is increasing in Λ_i^{-1}. To see this, assume $J > K$. This is equivalent to $\|KJ^{-1}\|_2 < 1$[21]. Then:

$$\left\| \left(K - \frac{1}{u'Ku} Kuu'K \right) \left(J - \frac{1}{u'Ju} Juu'J \right)^{-1} \right\|_2 =$$

[21]Cfr. e.g. Horn and Johnson (2012)

$$\left\| \left(I - \frac{1}{u'Ku} Kuu' \right) KJ^{-1} \left(I - \frac{1}{u'Ju} Juu' \right)^{-1} \right\|_2$$
$$\leq \| \left(I - \frac{1}{u'Ku} Kuu' \right) \|_2 \|_2 KJ^{-1} \|_2 \|_2 \left(I - \frac{1}{u'Ju} uu'J \right)^{-1} \|_2$$

Now $I - \frac{1}{u'Ku} Kuu'$ has one zero eigenvalue and all the others are 1[22], so $\|\left(I - \frac{1}{u'Ku} Kuu'\right)\|_2 = 1$ and similarly $\|\left(I - \frac{1}{u'Ju} uu'J\right)\|_2 = 1$. Finally, $\|KJ^{-1}\|_2 < 1$ by assumption, so it follows that

$$\left\| \left(K - \frac{1}{u'Ku} Kuu'K \right) \left(J - \frac{1}{u'Ju} Juu'J \right)^{-1} \right\|_2 < 1$$

so that $J - \frac{1}{u'Ju} Juu'J > K - \frac{1}{u'Ku} Kuu'K$ as I wanted to show. □

Convergence We are going to need the following lemma.

Lemma 1. If a sequence of *symmetric* matrices B_n is monotone in the positive semidefinite ordering, and bounded in the 2-norm, then it converges.

Proof. Consider the case that the sequence is decreasing, that is $B_n - B_{n+1}$ positive semidefinite. The increasing case is analogous. Assume by contraposition that it does not converge. Then since it is bounded, by compactness there exists a converging subsequence B_{n_k}. Then in particular this sequence is also Cauchy, so:

$$\forall \varepsilon \exists K_0 : k_1, k_2 > K_0 \Rightarrow \|B_{n_{k_1}} - B_{n_{k_2}}\|_2 < \varepsilon$$

But then for any $n, m > n_{K_0}$ by the fact that the sequence is decreasing we can find k_1, k_2 such that $B_{n_{k_1}} > B_n > B_m > B_{n_{k_2}}$. Now we can write:

$$B_{n_{k_1}} - B_{n_{k_2}} = B_{n_{k_1}} - B_n + B_n - B_m + B_m - B_{n_{k_2}}$$

and we know that $B_{n_{k_1}} - B_n + B_m - B_{n_{k_2}}$ is positive definite, hence the maximum eigenvalue of the right hand side must be larger than the maximum eigenvalue of $B_n - B_m$. But the maximum eigenvalue is the norm, so $\|B_n - B_m\|_2 \leq \|B_{n_{k_1}} - B_{n_{k_2}}\|_2$ which proves that the whole sequence is Cauchy and so converges.

□

Define:

$$BR_{i,n+1} = ([C_i^{-1}]_{-1} + ((n_i - 1) BR_{i,n} + [\overline{\Lambda}_i]_{-1})^{-1})^{-1}$$

I will prove that the sequnce $(BR_{i,n})_n$ with the proper initial conditions constitute a decreasing sequence in the positive semidefinite ordering. From this, the fact that it is bounded as proven in the existence theorem, and the previous lemma, it follows that they converge.

[22]Summing by the identity matrix results in all eigenvalues being shifted by 1, and $\frac{1}{u'Ku} Kuu'$ has rank 1 with eigenvalue 1, realized by eigenvector Ku.

From above Set $BR_{i,0} = C_{i,-1}$. I prove that $BR_{i,0} - BR_{i,1} = C_{i,-1}(\overline{\Lambda}_i + 2C_{i,-1})^{-1}C_{i,-1}$ and so is positive definite.

$$C_i - BR_{i,1} = C_{i,-1} - ((C_{i,-1} + \overline{\Lambda}_i)^{-1} + C_{i,-1}^{-1})^{-1} = C_{i,-1} - (C_{i,-1} - C_{i,-1}(C_{i,-1} + \overline{\Lambda}_i + C_{i,-1})^{-1}C_{i,-1})$$

$$= C_{i,-1}(2C_{i,-1} + \overline{\Lambda}_i)^{-1}C_{i,-1}$$

where the last but one step is by Woodbury formula. The matrix on the right hand side is positive definite because $(2C_i + \overline{\Lambda}_i)^{-1}$ is.

But then, since the best reply map is increasing when all matrices are symmetric, it follows that $B_{i,n} > B_{i,n+1}$ for each n, so the sequence is decreasing, which is what we wanted to show.

From below Now I prove that if $\tilde{B}_i$ has norm small enough, then $BR_i > \tilde{B}_i$. From this, and the fact that the best reply is increasing will follow convergence from below. Indeed:

$$BR_i > \tilde{B}_i \Leftrightarrow \|\tilde{B}_i BR_i^{-1}\|_2 < 1$$

and

$$\|\tilde{B}_i BR_i^{-1}\|_2 = \|\tilde{B}_i \left(C_{i,-1}^{-1} + (\overline{\Lambda}_i + (n_i - 1)\tilde{B}_i)^{-1} \right)\|_2 =$$

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$$\|\tilde{B}_i C_{i,-1}^{-1} + (\overline{\Lambda}_i \tilde{B}_i^{-1} + (n_i - 1)I)^{-1}\|_2 \leq \|\tilde{B}_i\|_2 \|(C_{i,-1}^{-1}\|_2 + \|(\overline{\Lambda}_i \tilde{B}_i^{-1} + (n_i - 1)I)^{-1}\|_2$$

Moreover:

$$\|(\overline{\Lambda}_i \tilde{B}_i^{-1} + (n_i - 1)I)^{-1}\|_2 = \frac{1}{\lambda_n((\overline{\Lambda}_i \tilde{B}_i^{-1} + (n_i - 1)I))} = \frac{1}{\lambda_n(\overline{\Lambda}_i \tilde{B}_i^{-1}) + (n_i - 1)I} < \frac{1}{n_i - 1}$$

where λ_n is the minimum eigenvalue of $(\overline{\Lambda}_i \tilde{B}_i^{-1} + (n_i - 1)I)$ and it is positive because it is positive definite.

Now if $\|\tilde{B}_i\|_2$ is smaller than $\frac{n_i - 1}{n_i}\|C_i^{-1}\|_2^{-1}$ it follows:

$$\|\tilde{B}_i BR_i^{-1}\|_2 < 1$$

which is what we wanted to show.

There exist a profile of coefficients implying positive trade The previous paragraph prove that a profile of matrix coefficients satisfying 16 esits. Now I prove that there exist one that yields positive trade if we limit ourselves to a subset of links - that will be the active links in equilibrium.

Start from the original network $G = (N, E)$. Set $n = 0$ and $L_1 = E$.

1. Find the unconstrained equilibrium profile B_n^* in the network $G_i = (N, L_n)$. Identify the set of links that have negative trade or negative price $E_{n,0}$.

2. set $L_{n+1} = L_n / E_{n,0}$;

The set of links shrink at each step, and when the network is empty there are no negative trades. Hence there must exist an index $\hat{\imath}$ such that for all $i > \hat{\imath}$ $L_i = L_{\hat{\imath}}$. The equilibrium $B^*_{\hat{\imath}}$, augmented with identically zero functions for all excluded links, is an equilibrium of the original game.

Generic Equilibrium existence It remains to prove that the profile of matrices $(B^*_i)_i$ identified above constitute the coefficient matrices of a profile of linear schedules for an open set $\mathcal{P} \times \mathcal{E}$ that contains $(p^*(0), 0)$. To prove this, consider the linear functions defined by $(B^*_i)_i$ and extend them to the whole price space. That is consider:

$$(S_{-1}, D)_i = \tilde{B}_i(-p_1\tilde{u}_{-1} + p_{i,-1} + t_i) + \varepsilon_i B_{\varepsilon,i,-1}$$

where t_i solves the Linear Complementarity problem:

$$\tilde{B}_i(-p_1\tilde{u}_{-1}+p_{i,-1}+t_i)+\varepsilon_i B_{\varepsilon,i,-1} \geq 0 \quad t'_{i,-1}(\tilde{B}_i(-p_1\tilde{u}_{-1}+p_{i,-1}+t_i)+\varepsilon_i B_{\varepsilon,i}) = 0 \quad t_i \geq 0$$

This corresponds to the form of the solution of the Optimization 41, where t_i is a function of the Lagrange multipliers on the nonnegativity constraints. Concavity proves that the solution is unique and so non-ambiguous.

Using this form we see that the market clearing conditions can be written as a Linear Complementarity Problem:

$$B_{ij}(p_i + t_i) + \varepsilon_i B_{\varepsilon,i} = B_{ij}(p_j + t_j) + \varepsilon_i B_{\varepsilon,j} \tag{45}$$

$$B_i(p_i + t_i) + \varepsilon_i B_\varepsilon \geq 0 \tag{46}$$

$$t'_i(B_i(p_i + t_i) + \varepsilon_i B_\varepsilon) = 0 \tag{47}$$

$$t_i \geq 0 \tag{48}$$

The first set of equations can be rewritten as $M(p + t) = \boldsymbol{A} + M_\varepsilon \varepsilon$ and solved for $p + t$ since M is invertible. So to compute which t variables are not zero it is sufficient to use the complementary slackness condition. Moreover, it is a standard result (Cottle et al. (2009), Proposition 1.4.6) that the solution as a function of ε is piecewise linear.

Now the fact that we can express the residual demand as a linear function for all i relies on the fact that $(0, p^*(0))$ lies in one of the regions where the function is linear and not on one of the boundary regions. Now the boundary regions are identified by a set of equations $F_j((B_i)_i, \varepsilon) = 0$ for some indices j, where the F are analytic functions (see Cottle et al. (2009), Prop. 1.4.6.). This means that the set of profiles of coefficients such that 0 is in one of the boundary regions:

$$\mathcal{B}_F = \{(B_i)_i \mid F_j((B_i)_i, 0) = 0\}$$

is rare. This follows from the fact that if there were an open set in $\mathcal{B}_F$ then since F is analytic it would have be identically zero, which it is not. Moreover $\mathcal{B}_F$ is closed, hence equal to its closure: hence its closure has empty interior, so it is rare.

Now consider the map $\mathcal{O} : (\omega_i)_i \to (B_i^*)$ that maps the values of the parameters to the B_i^* that solve 16. I prove that this is one-to-one. To see this, suppose $\mathcal{O}((\omega_i)_i) = \mathcal{O}((\omega_i')_i)$. Then by the construction of 2 we get that $\Lambda_i((\omega_i)_i) = \Lambda_i((\omega_i')_i)$, and by the equation 16 we get that the perfect competition matrices must agree too: $(C_i)_i = (C_i')_i$. From this, inspecting the matrix, it follows that $(\omega_i)_i = (\omega_i')_i$. Moreover it is continuous (actually analytic).

Since $\mathcal{O}$ si a homeomorphism the preimage of a rare set is rare, and so we conclude that the property of existence of a linear eqilibrium is generic in $(\omega_i)_i$.

B.2 Proof of Corollary 3.1

The fixed point equation 16 can be rewritten as:

$$(n_i - 1)\tilde{B}_i C_i^{-1} \tilde{B}_i + (\overline{\Lambda}_i C_i^{-1} + (n_i - 2)I)\tilde{B}_i - \overline{\Lambda}_i = 0 \tag{49}$$

and premultiplying by C_i^{-1}:

$$(n_i - 1)(C_i^{-1}\tilde{B}_i)^2 + (C_i^{-1}\overline{\Lambda}_i + (n_i - 2)I)(C_i^{-1}\tilde{B}_i) - C_i^{-1}\overline{\Lambda}_i = 0$$

Call $X = C_i^{-1}\tilde{B}_i$, $b = \frac{1}{n_i - 1}(C_i^{-1}\overline{\Lambda}_i + (n_i - 2)I)$ and $c = -\frac{1}{n_i - 1}C_i^{-1}\overline{\Lambda}_i$. We can rewrite this as:

$$b^2 - 4c = 4X^2 + 4bX + 4b^2$$

Now note that any solution of 49 commutes with b, because taking the transpose of the equation we get that X must solve also $X^2 + Xb + c = 0$, and so $-(X^2 + c) = bX = Xb$. Then the right hand side above is a square, and we have the analogous of the classical quadratic formula:

$$X = \frac{1}{2}\left(-b + \sqrt{b^2 - 4c}\right)$$

and so:

$$B_i = \frac{1}{2}C_i\left(-b + \sqrt{b^2 - 4c}\right)$$

Now $b^2 - 4c$ is the sum of two symmetric positive definite matrices, so is symmetric positive definite, and hence has a unique positive definite square root (Horn and Johnson (2012), Theorem 7.2.6). Hence the equation 49 has a unique positive definite solution, so the sector-level symmetric equilibrium is unique. □

B.3 Proof of Corollary 3.2

I omit the index i because all matrices are relative to sector i.

The quadratic labor cost of the profit is $\sum_{k,j}(\omega_{ij}\lambda_{ki} - \mu_{ij})^2$. This can be written in matrix form as $\begin{pmatrix} \lambda', & -\mu' \end{pmatrix} U'U \begin{pmatrix} \lambda \\ -\mu \end{pmatrix}$ where $U = [I_{out} \otimes \omega_i, u_{out} \otimes I_{in}]$.

Moreover $\begin{pmatrix} \lambda \\ -\mu \end{pmatrix} = V \begin{pmatrix} p_{out} \\ -p_{in} \end{pmatrix}$. and:

$$\pi = p'\left(B - \frac{1}{2}V'U'UV\right) = p'\left(B - \frac{1}{2}V'CV\right)$$

and:

$$V'CV = B\begin{pmatrix} 1 & \mathbf{0} \\ \tilde{u}_{-1} & C_{-1}^{-1} \end{pmatrix} C \begin{pmatrix} 1 & \tilde{u}'_{-1} \\ \mathbf{0} & C_{-1}^{-1} \end{pmatrix} B$$

since $C\tilde{u} = 0$ and $\tilde{u}'C = 0$. Moreover:

$$B\begin{pmatrix} 1 & \mathbf{0} \\ \tilde{u}_{-1} & C_{-1}^{-1} \end{pmatrix} C \begin{pmatrix} 1 & \tilde{u}'_{-1} \\ \mathbf{0} & C_{-1}^{-1} \end{pmatrix} B = \begin{pmatrix} \tilde{u}'_{-1}B_{-1}C_{-1}^{-1}B_{-1}\tilde{u}_{-1} & \tilde{u}'_{-1}B_{-1}C_{-1}^{-1}B_{-1} \\ B_{-1}C_{-1}^{-1}B_{-1}\tilde{u}_{-1} & B_{-1}C_{-1}^{-1}B_{-1} \end{pmatrix}$$

Now $B - V'CV$ has $\tilde{u}$ in the null space, and is positive semidefinite if and only if $B_{-1} - \frac{1}{2}B_{-1}C_{-1}^{-1}B_{-1}$ is. This is true because:

$$\|B_{-1}C_{-1}^{-1}B_{-1}B_{-1}^{-1}\|_2 = \|B_{-1}C_{-1}^{-1}\|_2 < 1$$

because we know that $B_{-1} < C_{-1}$.

B.4 Proof of Proposition 3

I am going to prove that, in any equilibrium, B has the following form: it is equal to $P\mathcal{B}P$, where $\mathcal{B}$ is an M-matrix (a positive definite matrix with positive diagonal and nonpositive off-diagonal entries), and:

$$P = \begin{pmatrix} I & \mathbf{0} \\ \mathbf{0} & -I \end{pmatrix}$$

is a matrix that changes signs to the off-diagonal blocks of $\mathcal{B}$.

From the definition of M in 2 it is immediate to see that, if B has the property above, then M is an M-matrix.

First, I prove that if a profile of coefficients $(B_i)_{i\in I}$ that has the property above then the best reply profile has still the property above.

By 2 we know that $\Lambda^{-1} = PLP$, where L is an M-matrix.[23] Moreover, in equilibrium, since $B\tilde{u} = 0$, we have $\Lambda^{-1}\tilde{u} = -\tilde{\boldsymbol{A}}_i \geq 0$. This is equivalent to $LP\tilde{u} = Lu > 0$, once we define $u = P\tilde{u}$. Then, we have that the matrix:

$$L - \frac{1}{u'Lu}Luu'L$$

[23] This follows because the proof in 2 shows that L is the Schur complement of an M-matrix, which is itself an M-matrix (see Horn et al. (1994)).

is positive semidefinite and still an M-matrix. Then it follows that also $\overline{\Lambda}^{-1}$ has the form $\overline{\Lambda}^{-1} = P\overline{L}P$ for an M-matrix $\overline{L}$, because:

$$\begin{aligned}\overline{\Lambda}^{-1} =& \Lambda^{-1} - \frac{1}{\tilde{u}'\Lambda^{-1}\tilde{u}}\Lambda^{-1}\tilde{u}\tilde{u}'\Lambda^{-1} \\ =& P(L - \frac{1}{u'Lu}Luu'L)P = P\overline{L}P\end{aligned}$$

(to get the expression, note that $P^2 = I$.)

Now, also the perfect competition matrix C has the same property: $C = P\mathcal{C}P$. Then, calling $J = \Lambda^{-1}+(n-1)B$, we can write the best reply equation as:

$$BR =$$

$$(I+(\overline{\Lambda}^{-1}+(n-1)B)^{-1}C)^{-1}(\overline{\Lambda}^{-1}+(n-1)B) = P(I+(\overline{L}+(n-1)\mathcal{B})^{-1}\mathcal{C})^{-1}(\overline{L}+(n-1)\mathcal{B})P$$

so that the best reply preserves the property.

To prove that any equilibrium profile has this property, proceed similarly to the proof of Theorem 1, step d). That is define $\tilde{B}_0$ as C, and consider the iteration:

$$\tilde{B}_n = (C_{-1} + ((\overline{\Lambda}_i^{-1})_{-1} + \tilde{B}_{n-1})^{-1})^{-1}$$

Notice that differently from Theorem 1 here $\overline{\Lambda}_i^{-1}$ is kept fixed. By an analogous argument this sequence is increasing and converges to the solution of the best reply equation, which is unique by Corollary 3.2. Moreover, each matrix of the sequence has the desired form, hence also the limit has. This is true because weak inequalities are preserved in the limit, and we already know that the limit is positive definite so it must have strictly positive diagonal. Hence it follows that any solution of the best reply equation must have the desired property.

□

C Proofs of Section 7

C.1 Proof of Proposition 6

I prove the result for a supply chain (line network) of length K. Denote the production function as f and the inverse demand at stage i of the chain as $P_i(\cdot)$. Assume both are differentiable and concave, $f' > 0$ and $P_i' < 0$. For every step of the chain but the first we have that firms optimize:

$$P_i(Q_i)q_{i\alpha} - p_{i-1}f^{-1}(q_{i\alpha})$$

where $Q_i = \sum_\alpha q_{i\alpha}$. By concavity they do so through the first order conditions[24]:

$$P_i'(Q_i)q_{i\alpha} + P_i(Q_i) - \frac{p_{i-1}}{f'(f^{-1}(q_{i\alpha}))} = 0$$

so in the symmetric equilibrium:

$$P_i'(Q_i)\frac{Q_i}{n_i} + P_i(Q_i) - \frac{p_{i-1}}{f'(f^{-1}\left(\frac{Q_i}{n_i}\right)} = 0$$

and the markup is determined by the usual elasticity condition:

$$\frac{p_i - MC_i}{p_i} = -\frac{P_i'(Q_i)\frac{Q_i}{n_i}}{P_i(Q_i)}$$

The equation allows to write directly the inverse demand that sector $i-1$ is facing (using the market clearing $Q_i = n_i f\left(\frac{Q_{i-1}}{n_i}\right)$) [25]:

$$P_{i-1}(Q_{i-1}) = \left[P_i'\left(n_i f\left(\frac{Q_{i-1}}{n_i}\right)\right) f\left(\frac{Q_{i-1}}{n_i}\right) + P_i\left(n_i f\left(\frac{Q_{i-1}}{n_i}\right)\right)\right] f'\left(\frac{Q_{i-1}}{n_i}\right)$$

To compare the elasticities, first calculate the derivative of this:

$$P_{i-1}'(Q_{i-1}) = \left[P_i''\left(n_i f\left(\frac{Q_{i-1}}{n_i}\right)\right) f'\left(\frac{Q_{i-1}}{n_i}\right) f\left(\frac{Q_{i-1}}{n_i}\right)\right.$$

$$\left. + \left(1 + \frac{1}{n_i}\right) f'\left(\frac{Q_{i-1}}{n_i}\right) P_i'\left(n_i f\left(\frac{Q_{i-1}}{n_i}\right)\right)\right] f'\left(\frac{Q_{i-1}}{n_i}\right) +$$

$$\left[P_i'\left(n_i f\left(\frac{Q_{i-1}}{n_i}\right)\right) f\left(\frac{Q_{i-1}}{n_i}\right) + P_i\left(n_i f\left(\frac{Q_{i-1}}{n_i}\right)\right)\right] \frac{1}{n_i} f''\left(\frac{Q_{i-1}}{n_i}\right)$$

By concavity, the first and last terms are negative, so we conclude:

$$P_{i-1}'(Q_i) < P_i'(f')^2\left(1 + \frac{1}{n_i}\right)$$

so

$$\frac{P_{i-1}'}{P_{i-1}}\frac{Q_{i-1}}{n_{i-1}} < \frac{P_i'(f')^2\left(1 + \frac{1}{n_i}\right)}{(P_i + P_i'f)f'}\frac{Q_{i-1}}{n_{i-1}}$$

$$= \frac{P_i'f'\left(1 + \frac{1}{n_i}\right)}{P_i + P_i'f}\frac{n_i}{n_{i-1}}\frac{Q_{i-1}}{n_i}$$

---

[24]The second derivative of the profit function is:

$$P_i''(Q_i)q_{i\alpha} + 2P_i'(Q_i) + \frac{p_{i-1}f''(f^{-1}(q_{i\alpha}))}{f'(f^{-1}(q_{i\alpha}))}\frac{p_{i-1}}{(f'(f^{-1}(q_{i\alpha})))^2}$$

By concavity of P_i and f this is negative.

[25] Differentiating this expression we immediately get that it is decreasing in Q_i.

moreover, we have that:

$$\frac{P_i' f' \left(1+\frac{1}{n_i}\right)}{P_i + P_i' f} \frac{n_i}{n_{i-1}} \frac{Q_{i-1}}{n_i} < \frac{P_i'}{P_i} \frac{Q_i}{n_i}$$

if and only if:

$$P_i P_i' f' \frac{Q_{i-1}}{n_i} \left(1+\frac{1}{n_i}\right) \frac{n_i}{n_{i-1}} < P_i P_i' \frac{Q_i}{n_i} + \left(P_i'\right)^2 \frac{Q_i}{n_i} \frac{Q_{i-1}}{n_i}$$

$$P_i P_i' \left(f' \frac{Q_{i-1}}{n_i} \left(1+\frac{1}{n_i}\right) \frac{n_i}{n_{i-1}} - \frac{Q_i}{n_i}\right) < (P_i' f)^2$$

Now if n_i and n_{i-1} are sufficiently close the parenthesis is positive by concavity of f (which implies $f' \frac{Q_{i-1}}{n_i} > \frac{Q_i}{n_i}$), hence the inequality is always satisfied. In particular this is true if $n_i = n_{i-1}$. We can conclude that in equilibrium if n_i and n_{i-1} are sufficiently close:

$$\frac{P_i'}{P_i} \frac{Q_i}{n_i} < \frac{P_{i-1}'}{P_{i-1}} \frac{Q_{i-1}}{n_{i-1}}$$

and so firms in sector $i-1$ have larger markup than firms in sector i.

For the case of markdowns, the exact analogous calculations hold, on supply rather than demand functions.

C.2 Proof of Propositions 4 and 5

The proofs follow from the following lemmas.

Lemma 2. The profile $\mathcal{B} = ((\frac{D_i S_i}{D_i + S_i})_{i \geq 2}, D_1)$ is a symmetric function of the "sector level" coefficients $(n_i B_i)$. That is $\mathcal{B}((n_i B_i)_i) = \mathcal{B}(n_{\pi(i)} B_{\pi(i)})_i$ where π is any permutation of indices.

Proof. By induction, I prove that $\mathcal{B}$ is equal to:

$$\frac{D_i^r S_i^r}{D_i^r + S_i^r} = \frac{\prod_{k \neq i} n_k B_k B_c}{\prod_{k \neq i} n_k B_k + B_c \sum_{j \neq i} \prod_{k \neq i, k \neq j} n_k B_k} \tag{50}$$

$$D_1 = \frac{\prod_{k \neq 1} n_k B_k B_c}{\prod_{k \neq 1} n_k B_k + B_c \sum_{j \neq 1} \prod_{k \neq 1, k \neq j} n_k B_k} \tag{51}$$

By induction on the size of the line N. If $N = 2$ it can be checked by calculation. Assume it holds for a line of size $N-1$. To get the corresponding expressions for a line of size N we must substitute B_c with the objective demand of the last but one layer, which is $\frac{n_N B_N B_c}{n_N B_N + B_c}$. If we do it we get that for $i \leq N-1$:

$$\frac{D_i S_i}{D_i + S_i} = \frac{\prod_{k \neq i} n_k B_k \frac{n_N B_N B_c}{n_N B_N + B_c}}{\prod_{k \neq i} n_k B_k + \frac{n_N B_N B_c}{n_N B_N + B_c} \sum_{j \neq i} \prod_{k \neq i, k \neq j} n_k B_k}$$

and reordering and simplifying the denominator we get the expression above. Analogously can be done for D_1. Moreover, always by induction we can find:

$$S_N = \frac{\prod_{k \neq N} n_k B_k}{\prod_{k \neq N} n_k B_k}$$

and $D_N = B_c$, so substituting in the corresponding expression:

$$\frac{D_N S_N}{D_N + S_N} = \frac{\frac{\prod_{k \neq N} n_k B_k}{\prod_{k \neq N} n_k B_k} B_c}{B_c + \frac{\prod_{k \neq N} n_k B_k}{\prod_{k \neq N} n_k B_k}}$$

and simplifying we get the desired result.

□

Lemma 3. In equilibrium $n_i > n_j$ implies $B_i^* > B_j^*$.

Proof. To apply the theory of monotone comparative statics, I will prove that if $n_i \geq n_j$ then $BR_i(x, B_{-i,j}) \geq BR_j(x, B_{-i,j})$, that is the best reply of i dominates the best reply of j conditional on the coefficients of all other sectors.

We have that $BR_i \geq BR_j$ if and only if:

$$\overline{\Lambda}_i^{-1} + (n_i - 1)x \geq \overline{\Lambda}_j^{-1} + (n_j - 1)x$$

In particular, using the characterization of $\overline{\Lambda}_i^{-1}$ above, we have that this is true if and only if:

$$\frac{\mathcal{B} B_c n_j x}{\mathcal{B}(n_j x + b_C) + n_j x \mathcal{F}} + (n_i - 1)x \geq \frac{\mathcal{B} B_c n_i x}{\mathcal{B}(n_i x + b_C) + n_i x \mathcal{F}} + (n_j - 1)x$$

where $\mathcal{B}$ and $\mathcal{F}$ are only functions of the coefficients $B_{-i,j}$ and their respective number of firms. This is true if and only if

$$\frac{\mathcal{B} B_c n_j}{\mathcal{B}(n_j x + b_C) + n_j x \mathcal{F}} - (n_j - 1) \geq \frac{\mathcal{B} B_c n_i}{\mathcal{B}(n_i x + b_C) + n_i x \mathcal{F}} - (n_i - 1)$$

$$\frac{\mathcal{B} B_c n_j - (n_j - 1)(\mathcal{B}(n_j x + B_C) + n_j x \mathcal{F})}{\mathcal{B}(n_j x + B_C) + n_j x \mathcal{F}} \geq \frac{\mathcal{B} B_c n_i - (n_i - 1)(\mathcal{B}(n_j x + B_C) + n_j x \mathcal{F})}{\mathcal{B}(n_i x + B_C) + n_i x \mathcal{F}}$$

$$\frac{\mathcal{B} B_c - (n_j - 1)(\mathcal{B}(n_j x) + n_j x \mathcal{F})}{\mathcal{B}(n_j x + b_C) + n_j x \mathcal{F}} \geq \frac{\mathcal{B} B_c - (n_i - 1)(\mathcal{B}(n_j x) + n_j x \mathcal{F})}{\mathcal{B}(n_i x + B_C) + n_i x \mathcal{F}}$$

which is true if and only if $n_i \geq n_j$ because the function is decreasing.

Then we can conclude that if $n_i \geq n_j$ then $BR_i(x, B_{-i,j}) \geq BR_j(x, B_{-i,j})$, and so, using a result from Lazzati (2013) we can conclude that in equilibrium $B_i^* \geq B_j^*$.

□

Proof of Proposition 4 Calculations reveal that:

$$M_i = p_i - \lambda_i = \frac{S_i^r}{(D_i^r + S_i^r)(1 + B_i) + S_i^r D_i^r}(p_i - p_{i-1})$$

$$= \frac{\frac{S_i^r}{D_i^r + S_i^r}}{(1 + B_i) + \frac{D_i^r S_i^r}{D_i^r + S_i^r}}(p_i - p_{i-1})$$

$$m_i = \mu_i - p_{i-1} = \frac{\frac{D_i^r}{D_i^r + S_i^r}}{(1 + B_i) + \frac{D_i^r S_i^r}{D_i^r + S_i^r}}(p_i - p_{i-1})$$

Now by the previous lemma $B_i = B_j$ for all sectors and so market clearing conditions imply that $p_i - p_{i-1}$ is constant across sectors. Moreover by lemma 2 also $\frac{D_i^r S_i^r}{D_i^r + S_i^r}$. Now inspecting the right hand side of the expressions we see that the markup is decreasing with D_i^r, which is itself decreasing as one goes upstream. Then it follows that the markup is increasing going upstream, and symmetrically for the markdown.

Proof of Proposition 5 We can rewrite the profits as:

$$\pi_i = \frac{1 - \frac{1}{2}B_i}{n_i^2 B_i} c^2$$

where c is the quantity consumed by the consumer. Now by lemma 3 we can conclude.

D Proofs of Section 4

D.1 Proof of Theorem 2

I am going to prove that the best reply function is increasing in the parameters n_i. By monotone comparative statics this implies that in the maximal equilibrium coefficients B are larger, which means that price impacts are smaller.

First, note that M is increasing in any n_i. Indeed, if $n' \geq n$:

$$M(n_i') - M(n_i) = \sum_m x_m'(n_m' - n_m)\hat{B}_m x_m \geq 0$$

Then by the calculations in the proof of 1 it means that the best reply function is increasing in n, which is what we wanted to show.

Now assume that the consumer only buys one "final" good. The vector c has a nonzero entry only in correspondence of the consumer price. This means that $A_c p_c = c'p = c'M^{-1}c$. Since M is increasing in each B_i is also decreasing in each Λ_i, so it follows that the consumer price is increasing in Λ_i. $\square$

E Proofs of Section 5

E.1 Proof of Theorem 3

The best reply matrix for each firm in sector i is:

$$\left([C_i^{-1}]_{-1} + ((n_i - 1)\tilde{B}_i + B_i^D)^{-1}\right)^{-1} \tag{52}$$

where $\overline{B}_i$ is the diagonal matrix that on the diagonal has the coefficient $B_{k,ii}$ for all the neighbors k of i.

Let us define two functions, corresponding to the best reply in the global and local game:

$$BR_i(\tilde{B}_{-i}, g) = \left([C_i^{-1}]_{-1} + ((n_i - 1)\tilde{B}_i + \overline{\Lambda}_i^{-1})^{-1}\right)^{-1} \tag{53}$$

$$BR_i(\tilde{B}_{-i}, l) = \left([C_i^{-1}]_{-1} + ((n_i - 1)\tilde{B}_i + B_i^D)^{-1}\right)^{-1} \tag{54}$$

$$\tag{55}$$

The equilibrium profiles of matrix coefficients in the local or global equilibrium satisfie:

$$\tilde{B}_i^g = BR_i(\tilde{B}_{-i}^g, g) \tag{56}$$

$$\tilde{B}_i^l = BR_i(\tilde{B}_{-i}^l, l) \tag{57}$$

The result follows by applying the theory of monotone comparative statics. In particular, fixing the profile of matrices $\tilde{B}_{-i}$ by immediate application of the definition, we have that:

$$\overline{\Lambda}_i^{-1} \leq B_i^D \text{ in the psd ordering}$$

and so:

$$BR_i(\tilde{B}_{-i}^g, g) \leq BR_i(\tilde{B}_{-i}^l, l)$$

Then, we can apply the theory of monotone comparative statics, considering g or l the parameter, and considering an ordering on the parameter space such that $l \succ g$. Then the best reply equation is increasing in this parameter.

By standard arguments now we can conclude that in the maximal equilibrium $\tilde{B}_i^l \geq \tilde{B}_i^g$, and from this it follows that $M^l \geq M^l$, and so the final price

$$p_c^l = \boldsymbol{A}'(M^l)^{-1}\boldsymbol{A} \leq \boldsymbol{A}'(M^g)^{-1}\boldsymbol{A} = p_c^g$$