Low Competition Traps*

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Abstract

We study the complementarity between aggregate demand and competition in a multi-industry model with oligopolistic competition. Larger aggregate demand increases the degree of competition in product markets (as more firms enter, profit rates decrease and factor shares increase). Higher competition, on the other hand, boosts aggregate demand (higher factor shares increase factor supply and output). Such positive interaction between aggregate demand and competition may give rise to multiple steady-states. Negative transitory shocks may hence push the economy to episodes of persistently lower output and less intense competition (*low competition traps*). We show that the likelihood that the economy enters a *low competition trap* can increase with the degree of within-industry markup dispersion.

The model can rationalize important features of the US post-2008 growth experience. In particular, it explains how a long-run increase in markup dispersion, as documented by the literature, may have contributed to the severity of the recession and resulted in (i) a persistent deviation of per capita GDP from trend, (ii) a decline in the investment rate, (iii) a decline in interest rates, (iv) a fall in labor supply and (v) a significant drop of the labor share.

JEL Classification: E22, E24, E25, E32, L16

Key words: competition, market power, multiple equilibria, poverty traps, great recession

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1 Introduction

The 2008 recession seems to have had a persistent impact on a number of US macroeconomic aggregates, perhaps most notably on GDP per capita. As shown in Figure 1, since 2008 real GDP per capita has experienced a persistent (and increasing) deviation from its pre-crisis trend.[1] Such persistent deviation in the aftermath of a recession is unique in the US postwar experience.[2] Indeed, even after the Great Depression of 1929, and despite a larger initial contraction, real GDP per capita rebounded to the pre-crisis trend within one decade (see Figure 17 in Appendix A).

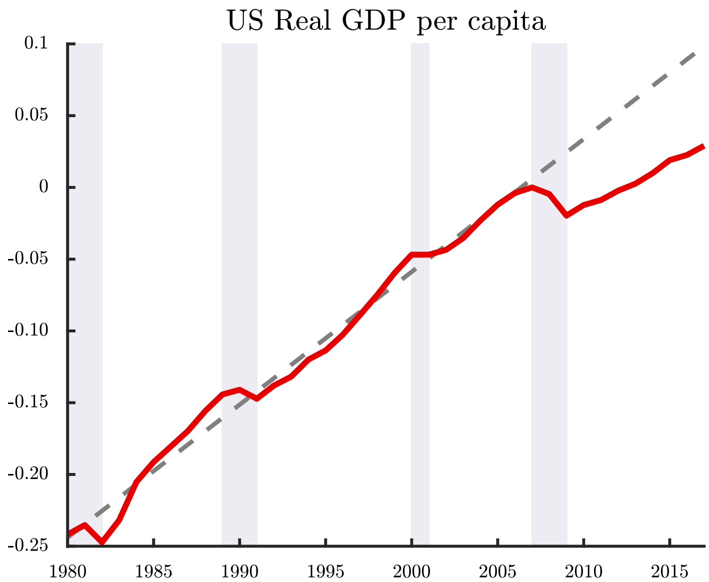


Figure 1: **US Real GDP per capita: 1980-2017**
Data is from the US Bureau of Economic Analysis. The series shown is on logs, undetrended and centered around 2007. The linear trend is computed for the 1980-2007 period.

The purpose of this paper is to provide a theory that can account for the persistent deviation documented in Figure 1 and discuss a possible reason why the 2008 crisis, contrarily to previous recessions, may have had such long-lasting effect. To this end we build a multi-industry model with oligopolistic competition and variables markups. At the heart of our model is the positive interaction between aggregate demand and competition. Larger aggregate demand increases the degree of competition in product markets (as more firms enter, profit rates decrease and factor shares increase). Higher competition, on the other hand, boosts aggregate demand (higher factor shares increase factor supply and output). Such positive interaction between aggregate demand and competition may give rise to multiple steady-states. In particular, there can be high steady-states where intense competition, by generating high factor prices, sustains high factor supply and high aggregate demand/output; and low steady-states where weak competition depresses factor prices and, through this channel, factor supply and factor demand. We refer to such low steady-states as *low competition traps*.

[1] The deviation has been not only persistent, but also large in magnitude: in 2017, US real GDP per capita was 7 log points below its pre-crisis trend. Given that US per capita GDP was roughly $60.000 in 2017, this fact implies a per capita income loss of $4.200.

[2] A persistent deviation from trend can be also found if we consider the postwar period 1947-2017 (see Figure 16 in Appendix A).

Negative transitory shocks may hence trigger a persistent transition from a high steady-state to a *low competition trap*. Importantly, we show that the likelihood that the economy enters a *low competition trap* can increase with the degree of within-industry markup dispersion.

To exemplify the mechanism, think of an economy in which two firms can operate in each industry: a productive/large firm and an unproductive/small firm. Since in our model there is a tight connection between productivity/size and markups, the large firm will have high market power and charge a high markup, whereas the small firm will have reduced market power and charge a lower markup. Note that, because of fixed costs of production, the second firm will only enter when it makes sufficiently large profits. This fact translates into a threshold of aggregate demand below which it stays out of the market. We show that such threshold is likely to increase with the level of markup dispersion. A more dispersed markup distribution means that, for any level of aggregate demand, the first firm is larger and makes larger profits, and the second firm is smaller and makes lower profits. As a result, the second firm can only survive at increasingly higher levels of aggregate demand. Within our model, this fact implies that a high steady-state in which the two firms operate is more difficult to be sustained and the economy is increasingly likely to fall in a steady-state in which only the first firm operates (i.e. a *low competition trap*). This is indeed an important take-away from our model - any force leading to an increase in markup dispersion can make the economy more fragile and more likely to end up in a lower steady-state.

Note that whenever the number of competing firms remains unchanged, a reallocation of activity from the unproductive/small firm towards to productive/large firm will be typically efficient. Therefore, in our model, an increase in markup dispersion is not detrimental *per se*. Markup dispersion entails a risk insofar as it makes the small/unproductive firm more likely to exit the market, thereby reducing the competitive pressure it poses on the market leader.

Our model has been motivated by two main sets of facts. The first empirical observation is that there is growing evidence of an upward trend in measures of firm-level heterogeneity, since at least the 1980s. For instance, De Loecker, Eeckhout and Unger (2018) have recently documented a steady increase in the variance of the markup distribution for US public firms since the 1980s (see Figure 22 and 23 in Appendix D).[3] Some authors have also found evidence of rising productivity dispersion. For instance, Kehrig (2015) and Decker et al. (2017) use data from the US census to document an upward trend in measures of (within-industry) productivity dispersion.[4] We show that these two facts may be tightly connected - since in our model there is a positive association between productivity and markups, an increase in productivity dispersion will be translated into a larger markup dispersion. An important contribution of our paper is to explain how a larger markup dispersion can make the economy more fragile and more likely to end up in a lower steady-state. Through the lens of our model, the persistent deviation of real GDP per capita from trend in 2008 can be

[3]This fact has translated into a steady increase in size-weighted average markups. In particular, the sales-weighted average of markup of US publicly listed firms increased from 1.21 in 1980 to 1.61 in 2016.

[4]Such finding seems consistent with Andrews, Criscuolo and Gal (2015), who document a widening gap between firms at the top and at the bottom of the (global) productivity distribution in a sample of OECD countries. See Van Reenen (2018) for a summary of the recent evidence that suggests an increase in firm-level heterogeneity.

linked to the long-run increase in markup dispersion that has taken place since the 1980s.

The second set of facts that motivate our model concerns the evolution of product market competition after the 2008 crisis. There are some signs that suggest a permanent increase in market power in the aftermath of the 2008 crisis. Figure 2 shows two indicators that can be related to the aggregate level of product market competition: (i) the number of active firms and (ii) the aggregate profit share (for each series, the grey dashed line represents the average over the 1980-2017 period). As one can see in the first panel of Figure 2, the number of active firms (per 1000 people) experienced a significant and persistent decline after 2008: between 2011 and 2016 there were on average 15.8 firms per 1000 people, against an average of 16.8 firms in the 1988-2007 period (such average was17.1 between 2000 and 2007). Such trends can also be detected within all major sectors of activity (see Figure 19 in Appendix C.1). The aggregate profit share, on the other hand, has been on average 10.6% between 2015 and 2017, against an average of 8.1% percent between 1980 and 2007. The two indicators shown in Figure 2 suggest that the level of product market competition may have experienced a persistent decline in the wake of the 2008 recession.

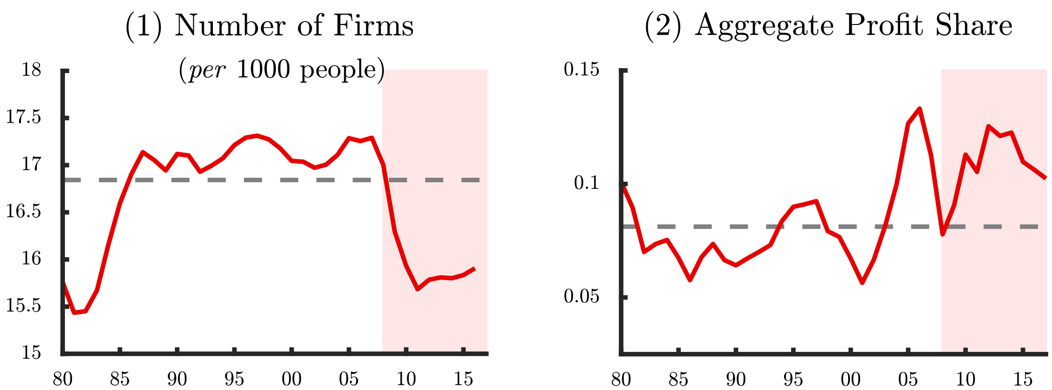


Figure 2: **Motivating Facts for the United States: 1980-2017**
(i) The first panel shows the number of firms with at least one employee (from the BDS) per 1000 people (ii) The aggregate profit share is ratio of domestic corporate profits to gross value added for the US private business sector (from the BEA). For each series, the dashed grey line shows the respective average for the 1980-2007 period.

Our theory also makes makes predictions on the evolution of factor shares, factor prices and factor demand/supply. In particular, as the economy enters a *low competition trap* we should observe a reduction in factor shares and factor prices, as well as a reduction in factor demand/supply. There is indeed evidence that supports these predictions. For instance, Gutiérrez and Philippon (2017) and Jones and Philipon (2017) have pointed out that the US investment rate has been low after 2008, in spite of historically low interest rates. This fact suggests a decline in firms' investment demand. Other authors have also noted that the 2008 recession has been associated with a persistent decline in labor market participation (see Brinca et al. (2016) and Fernald et al. (2017)) and with the minimum value of the US labor share in the postwar period (Elsby, Hobijn and Şahin (2013)).

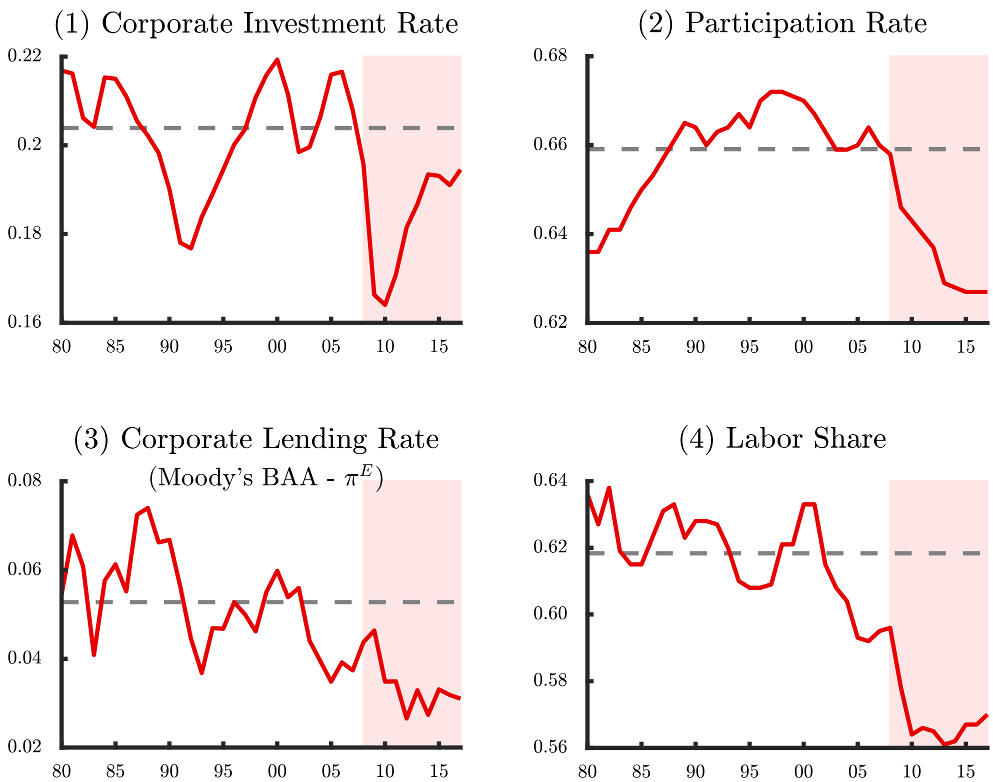


Figure 3: **Motivating Facts for the United States: 1980-2017**
(i) The investment rate is the ratio of gross capital formation to gross value added, for the US private business sector (from the BEA) (ii) The labor market participation rate is the percentage of the civilian noninstitutional population 16 years and older that is working or actively looking for work (from the BLS) (iii) The corporate lending rate is Moody's Seasoned BAA Corporate Bond Yield (from Moody's) minus a 3-year moving average of past CPI inflation (from the BLS) (iv) The labor share is ratio of total worker compensation to nominal output, for the US nonfarm business sector (from the BLS). For each series, the dashed grey line shows the corresponding average for the 1980-2007 period.

Figure 3 illustrates these facts. Panels (1) and (3) represent the corporate investment and interest rates, whereas panels (2) and (4) show labor market participation (or labor supply) and the labor share. For instance, the investment rate experienced a significant decline between 2008 and 2010 (of more than 4 percentage points). It has then increased between 2011 and 2014, but seems to have stabilized at a lower level: the average investment rate between 2015 and 2017 was 19.3%, against an average of 20.4% between 1980 and 2007. Panel (3) suggests that the real corporate lending rate has been below their pre-crisis average (the average interest rate between 2015 and 2017 was 3.20%, while the 1980-2007 average was 5.28%). Labor market participation also suffered a persistent decline in the wake of the 2008 recession. It has then stabilized at an average of 62.7% between 2015 and 2017, against an average of 65.9% between 1980 and 2007 (and of 66.5% between 1990 and 2007). Panel 4 shows that the aggregate labor share has experienced a substantial decline after 2008. Between 2015 and 2017, the fraction of US GDP accruing to labor was on average 56.8%, well below the 1980-2007 average of 61.8%.[5] As mentioned above, our theory can account for these four facts.

[5] There is evidence of a long-run downward trend in the labor share (see Karabarbounis and Neiman (2013)). However, even if we compute a linear trend for the labor share between 1980 and 2007, we still observe a significant and persistent deviation after 2008. See Figure 18 in Appendix B.

We shall refer that the current version of our model, while admitting a full analytical characterization and qualitatively matching the facts represented in Figures 1 to 3, is still not suitable for a realistic quantitative evaluation and a welfare analysis. We are extending the model to quantitatively assess the mechanism that we propose and to conduct a number of policy experiments (such as the introduction of firm entry subsidies). We also want to analyze welfare. Note that in our model the transition to a *low competition trap* does not necessarily entail a welfare loss - there is a natural cost associated with the loss of output, but also some benefits stemming from the resources that the economy saves as the unproductive/small firms exit the market.

2 Related Literature

Our paper can be related to three different strands of the literature. In first place, our paper belongs to the macroeconomic literature studying the cyclical properties of competition and markups, which includes the contributions of Rotemberg and Saloner (1986), Chatterjee, Cooper and Ravikumar (1993), and Gilchrist et al. (2017) among others. Close to our approach, Chatterjee, Cooper and Ravikumar (1993) build a multi-industry model with Cournot competition and an endogenous number of firms. Their model features a complementary between aggregate demand and firm entry decisions, which is capable of generating multiple equilibria and multiple steady-states. Compared to their work, we contribute by analyzing the aggregate implications of changes in the distribution of firm-level productivities and markups.

Second, this paper relates to a large and growing literature documenting long-run trends in firm heterogeneity and market power. In our model, an increase in productivity/markup dispersion will be associated with an increase in market power - since there is a positive association between size and markups, a more dispersed markup distribution translates into an increase in aggregate markups. De Loecker, Eeckhout and Unger (2018) estimate price-cost markups from financial statements for US public firms. They document a steady increase in the sales-weighted average markup since 1980 and 2016.[6] As the authors show, such tendency is closely associated with a shift in the distribution of markups. Using data from national accounts, Hall (2018) finds that the average US markup, weighted by value-added, increased from 1.12 in 1988 to 1.38 in 2015. Autor et al. (2016) document an increase in industry concentration in different sectors of activity. They show that, within each sector, industries with a faster increase in concentration, experienced a larger decline in the labor share. They are other signs suggesting a secular increase in market power. For example, Decker et al. (2014) document a secular decline in measures of business dynamism. Decker et al. (2017) and Kehrig and Vincent (2018) use firm level data from the US census to show that firms have become increasingly less reactive to their own TFP shocks. Such decline in firm responsiveness to TFP shocks is also consistent with increased market power.

Lastly, this paper also relates to the literature focusing on the persistent deviation of the US real GDP per capita from

[6]Edmond, Midrigan and Xu (2019) show that cost (as opposed to sales) weighted average markup display a less pronounced upward trend. In their setting, a cost-weighted average markup is the one that is relevant for welfare analysis, as it accounts for the fact that high markup firms are also more productive. See also Karabarbounis and Neiman (2018) and Traina (2018) for a critique on the De Loecker, Eeckhout and Unger (2018) methodology.

its pre-crisis trend. Schaal and Taschereau-Dumouchel (2015) study a model with endogenous capacity utilization. Their model features a complementary between firms' capacity utilization decisions and aggregate demand, which can give rise to equilibrium multiplicity and multiple steady-states. Like us, they interpret the deviation of real GDP per capita from trend as a transition to a low steady-state. Some other authors have proposed explanations based on endogenous R&D. Benigno and Fornaro (2018) study an economy featuring nominal rigidities and endogenous innovation. In their setting, there is a positive feedback between aggregate demand and firms' R&D decisions, which can also generate equilibrium multiplicity. In particular, the economy can enter prolonged periods of weak growth and where the zero lower bound on interest rates is binding. Anzoategui et al. (2018) also propose a model with endogenous R&D where a temporary contraction in aggregate demand can affect the production and diffusion of R&D. Finally, Guerron-Quintana and Jinnai (2018) build a model featuring financial frictions and endogenous R&D. In their model, adverse financial disturbances can have an impact on the economy's long run trend. While we see our theory as complementary to the above-mentioned articles, we think that we are the first to establish a link between the persistency of the 2008-recession and the long-run increase in firm level heterogeneity taking place since the 1980s.

3 The Model

3.1 Demographics and Preferences

Time is discrete and indexed by $t = 0, 1, 2, \ldots$. The economy consists of T overlapping generations. Individuals live for T periods: they are active for T_A periods and then retire for an additional T_R periods ($T = T_A + T_R$).

Agents differ in their rate of time preferences. We assume that, when an agent is born, he draws his discount rate β_i from a Pareto distribution with support $[\beta_m, \infty)$ and shape parameter γ, i.e.[7]

$$\Pr\left[\beta_i < \beta\right] = 1 - \left(\frac{\beta_m}{\beta}\right)^{\gamma} \quad , \quad \beta \geq \beta_m$$

Such discount rate will be fixed throughout the individual's lifetime. An agent with discount rate β_i and who is born at time t has utility

$$U_{i,t} = \mathbb{E} \sum_{s=0}^{T-1} \beta_i^s \, u\left(c_{i,t+s}, l_{i,t+s}\right)$$

The period utility function is

$$u\left(c_{i,t}, l_{i,t}\right) = \frac{1}{1-\delta} \left(c_{i,t} - \eta \frac{l_{i,t}^{1+\nu}}{1+\nu} - c_f n_{i,t}\right)^{1-\delta}$$

where $0 \leq \delta \leq 1$ and $\nu > 0$. Denoting by $a_{i,t}$ the assets of individual i at time t and by $T_{i,t}$ any lump sum transfer he

[7] The choice of a Pareto distribution is made for analytical convenience, as shall be clear below.

receives (for example, the profits generated in any firm he has created), we have the following dynamic budget constraint

$$a_{i,t} = W_t l_{i,t} + T_{i,t} + R_t a_{i,t-1} - c_{i,t}$$

We will impose the constraint $a_{i,t} \geq 0$ (i.e. individuals cannot have negative assets). This assumption will simplify some of the derivations, though it is not crucial for the main results.

Before proceeding, we shall justify the choice of the demographic structure. We chose a heterogeneous agent framework because we want to have variable steady-state interest rates (to match some of the facts discussed in Section 1). Recall that in a representative agent model, steady-state interest rates are pinned down by the agent's discount factor. Heterogeneity is achieved in this model through two channels: the overlapping generations structure and the heterogeneity in discount factors. Each feature alone would be sufficient to generate variable steady-state interest rates. The two features, while not unrealistic, will however reinforce each other.

Labor supply Each agent's labor supply will be a function of the aggregate wage rate

$$l_{i,t} = \begin{cases} \left(\dfrac{W_t}{\eta}\right)^{\dfrac{1}{\nu}} & \text{if } i \text{ is active} \\ 0 & \text{if } i \text{ is retired} \end{cases}$$

Consumption/savings decision

The consumption Euler equation is given by

$$\left(c_{i,t} - \eta \frac{l_{i,t}^{1+\nu}}{1+\nu}\right)^{-\delta} = \beta_i R_{t+1} \left(c_{i,t+1} - \eta \frac{l_{i,t+1}^{1+\nu}}{1+\nu}\right)^{-\delta}$$

This first order condition will establish a positive relationship between the savings rate and the interest rate, provided that $\delta < 1$. Such relationship can be clearly seen with the help of an example.

Example. Assume for simplicity that $\eta = 0$ and that $a_{i,t+1} = l_{i,t+1} = 0$. In such case, the individual savings rate $s_{i,t} := \dfrac{a_{i,t}}{W_t l_{i,t} + T_{i,t} + R_t a_{i,t-1}}$ is equal to

$$s_{i,t} = \frac{1}{\beta_i^{-\frac{1}{\delta}} R_{t+1}^{-\frac{1-\delta}{\delta}} + 1}$$

From the above expression, we can see that the savings rate is increasing in the interest rate R_{t+1}, provided that $\delta \in [0, 1)$. When $\delta = 1$ (log utility), the individual savings rate is independent of R_{t+1} and equal to $\dfrac{\beta_i}{1+\beta_i}$. On the other extreme, as $\delta \to 0$ (linear utility) we have that

$$\lim_{\delta \to 0} s_{i,t} = \begin{cases} 1 & \text{if } \beta_i R_{t+1} \geq 1 \\ 0 & \text{if } \beta_i R_{t+1} < 1 \end{cases}$$

i.e. agents with $\beta_i R_{t+1} \geq 1$ save all their wealth while agents with $\beta_i R_{t+1} < 1$ consume all their wealth.

We will assume for simplicity that $\delta \to 0$ (linear utility). Such limit case will allow us to have an elastic savings supply and yet keep the model highly tractable. In such case, the economy's aggregate savings rate coincides with the fraction of savers, i.e. the fraction of individuals who have $\beta_i \geq \dfrac{1}{R_{t+1}}$.

$$s_t = (\beta_m R_{t+1})^{\gamma} \tag{1}$$

γ captures the elasticity of the aggregate savings rate with respect to the interest rate.

3.2 Technology

There is a final good Y_t, which is a CES composite of different intermediate varieties

$$Y_t = \left(\int_0^1 y_{i,t}^{\rho} \, di \right)^{\frac{1}{\rho}}$$

where $y_{i,t}$ is the quantity of variety $i \in [0,1]$, $0 < \rho < 1$ and $\sigma \equiv \frac{1}{1-\rho}$ is the elasticity of substitution. The parameter ρ measures the degree of substitutability across varieties.[8] The final good is produced in a competitive sector and will be used as the *numeraire*.

The inverse demand for each good i is given by

$$p_{i,t} = \left(\frac{Y_t}{y_{i,t}} \right)^{1-\rho} \tag{2}$$

Each good i can be producer by different entrepreneurs, who are denoted by j. Entrepreneur j can produce variety $i \in [0,1]$ by combining capital $k_{i,t}^{j}$ and labor $l_{i,t}^{j}$ through a Cobb-Douglas technology

$$y_{i,t}^{j} = e^{\mu_t} \, \pi^{j} \left(k_{i,t}^{j} \right)^{\alpha} \left(l_{i,t}^{j} \right)^{1-\alpha} \tag{3}$$

According to this specification, the productivity of each entrepreneur depends on two terms: (i) a time-varying aggregate component e^{μ_t} (common to all industries and types) and a time-invariant idiosyncratic term π^{j}. μ_t is a white-noise process that satisfies

$$\mu_t \sim \mathbb{N}\left(0, \sigma_{\mu}^2\right) \tag{4}$$

We will let $\mathcal{F}_{\pi} := \left\{ \pi^1, \pi^2, \pi^3, \ldots \right\}$ denote the distribution of idiosyncratic productivity terms (which is assumed to be identical across industries). With no loss of generality, we will let type $j = 1$ have the highest productivity, type $j = 2$

[8] As $\rho \to 0$, final output becomes a Cobb-Douglas aggregate of intermediate varieties; in such case, production requires a strictly positive amount of each variety and the degree of differentiation is high. As $\rho \to 1$, intermediate varieties become perfect substitutes and the degree of differentiation vanishes.

have the second highest productivity, and so on

$$\pi^1 > \pi^2 > \pi^3 > \cdots$$

Labor is hired at the competitive wage W_t. The production of one unit of capital requires one unit of the final good. Furthermore, we shall assume that capital needs to be invested one period ahead and fully depreciates in production. Each unit of capital therefore costs R_t. Given these assumptions, entrepreneur j can produce good i with constant marginal cost $\frac{\Theta_t}{\pi^j}$, where

$$\Theta_t \equiv \left(\frac{R_t}{\alpha}\right)^{\alpha} \left(\frac{W_t}{1-\alpha}\right)^{1-\alpha}$$

is the marginal cost function for a Cobb-Douglas technology with unit productivity. We will often refer to Θ_t as the *factor cost index*.

In addition to all variable costs, the production of each variety entails a fixed production cost $c_f > 0$ per period. Such cost corresponds to an utility loss associated with management tasks.

3.3 Market Structure

Firms compete a la Cournot: all firms that decide to enter (thus incurring the fixed cost c_f) will simultaneously announce quantities, taking the output of the other competitors as given. We will assume that firms make sequential entry decisions in reverse order of productivity.[9]

Therefore, if entrepreneur j decides to produce in industry i, he will choose the amount of output $y_{i,t}^j$ that maximizes his profits given the output of his competitors. Specifically, he will solve

$$\max_{y_{i,t}^j} \quad \left(p_{i,t} - \frac{\Theta_t}{\pi^j}\right) y_{i,t}^j \qquad \text{s.t.} \quad p_{i,t} = \left(\frac{Y_t}{y_{i,t}}\right)^{1-\rho}$$
$$y_{i,t} = \textstyle\sum_{k=1}^{n_{i,t}} y_{i,t}^k$$

The solution to this problem yields a system of $n_{i,t}$ first order conditions

$$p_{i,t}\left[1 - (1-\rho)\, s_{i,t}^j\right] = \frac{\Theta_t}{\pi^j}$$

[9] When the fixed cost c_f is non-negligible and only a limited set of firms can profitably produce, there can be multiple equilibria. Suppose for instance that there are two types of firms: low productivity types with $\pi^j = \pi^L$ and high productivity types with $\pi^j = \pi^H > \pi^L$. Suppose further that demand and factor prices are such that any firm can profitably produce alone, but not if there is another competitor. In such case, there are two possible equilibria: a monopoly with a high type or a monopoly with a low type. Sequential entry in reverse order of productivity is a way to select a particular equilibrium (in this case, the monopoly with a high type). See Atkeson and Burstein (2008) for an identical assumption.

where $s^j_{i,t} \equiv \dfrac{y^j_{i,t}}{y_{i,t}}$ is the market share of firm $j = 1, \ldots, n_{i,t}$. As we can see from the previous equation, entrepreneur j will charge a markup $\left[1-(1-\rho)\, s^j_{i,t}\right]^{-1}$ over his marginal cost $\dfrac{\Theta_t}{\pi^j}$. There is hence a positive relationship between market shares and markups: large firms (which will be the most productive ones) charge higher markups.

We can solve the system above an obtain the equilibrium industry price. When the $n_{i,t}$ most productive firms produce, we have that

$$p_{i,t} = \frac{\sum_{k=1}^{n_{i,t}} \frac{1}{\pi^k}}{n_{i,t} - (1-\rho)} \Theta_t \tag{5}$$

Each firm j will have a market share

$$s^j_{i,t} = \frac{1}{1-\rho} \left[1 - \frac{n_{i,t} - (1-\rho)}{\sum_{k=1}^{n_{i,t}} \frac{1}{\pi^k}} \frac{1}{\pi^j} \right] \tag{6}$$

We only need to determine the equilibrium number of firms, $n_{i,t}$. The equilibrium number of firms must be such that (i) the profits $\left(p_{i,t} - \dfrac{\Theta_t}{\pi^j}\right) y^j_{i,t}$ of each active firm are not lower than the fixed cost c_f and (ii) if an additional firm where to enter, its profits would be lower than the fixed cost. To determine $n_{i,t}$, we follow an iterative procedure. We start with $n_{i,t} = 1$ and compute the equilibrium where firm π^1_i is the only producer (monopoly). We check whether the monopoly profits $\left(p_{i,t} - \dfrac{\Theta_t}{\pi^1}\right) y^1_{i,t}$ are larger than the fixed cost c_f. If they are, we let the second firm enter and compute the new equilibrium. We calculate the profits $\left(p_{i,t} - \dfrac{\Theta_t}{\pi^2}\right) y^2_{i,t}$ of the second firm. If they are larger than the fixed cost c_f, we let the third firm enter. We repeat the procedure until we find an equilibrium in which the profits of one additional firm are lower than the fixed cost, i.e. $\left(p_{i,t} - \dfrac{\Theta_t}{\pi^{n+1}}\right) y^{n+1}_{i,t} < c_f$.

The next two examples illustrate how the equilibrium is determined.

No productivity differences Suppose all firms are equally productive, so that $\pi^j = 1 \ \forall j$. In such case, each firm's market share will be simply the inverse of the number of active firms, $n_{i,t}$

$$s^j_{i,t} = \frac{1}{n_{i,t}}$$

The price will consist of a markup $\mu_{i,t} := \dfrac{n_{i,t}}{n_{i,t} - (1-\rho)}$ over the constant marginal cost Θ_t

$$p_{i,t} = \frac{n_{i,t}}{n_{i,t} - (1-\rho)} \Theta_t$$

The markup (and hence the price) decrease with the number of active firms. Note that the markup ranges from $\mu_{i,t} = \dfrac{1}{\rho} > 1$ when $n_{i,t} = 1$ (monopoly) to $\mu_{i,t} = 1$ when $n_{i,t} \to \infty$ (perfect competition).

Finally, note than when there are $n_{i,t}$ producers, each firms makes profits (exclusive of the fixed production cost)

$$\Pi(n_{i,t}, \Theta_t, Y_t) := \begin{cases} 0 & \text{if } n_{i,t} = 0 \\ \dfrac{1-\rho}{n_{i,t}^2} \left[\dfrac{n_{i,t} - (1-\rho)}{n_{i,t}}\right]^{\frac{\rho}{1-\rho}} \Theta_t^{-\frac{\rho}{1-\rho}} Y_t & \text{if } n_{i,t} \geq 1 \end{cases} \tag{7}$$

The function above can be easily shown to be decreasing in $n_{i,t}$. The equilibrium number of followers must be such that (i) all firms that operate do not make a loss, (ii) but if an additional firm were to enter it would incur a loss. Denoting by $n_{i,t}^*$ the equilibrium number of firms, we have that

$$\left[\Pi\left(n_{i,t}^*, \Theta_t, Y_t\right) - c_f\right] \left[\Pi\left(n_{i,t}^* + 1, \Theta_t, Y_t\right) - c_f\right] \leq 0$$

It follows immediately from (7) that profits, and hence the equilibrium number of firms, necessarily increase with aggregate output/demand Y_t.

Figure 4 below illustrates how the number of firms is determined. It also shows how a decrease in aggregate demand Y_t affects the equilibrium number of firms. In the example considered, when Y_t is high (blue line) there will be two firms producing. However, when Y_t is low (red line) the market can only accommodate one producer: if a second firm were to enter, its profits would be below the fixed cost of production c_f.

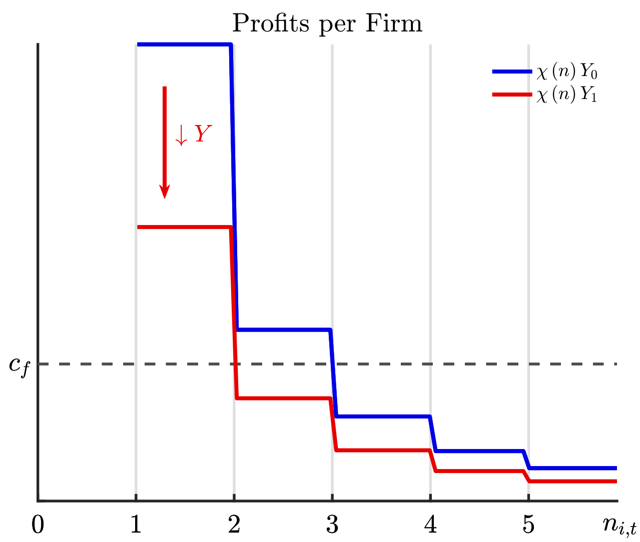


Figure 4: Profits per firm

One leader *versus* multiple followers Let us introduce a modification in the previous example. Suppose now that there is a productive firm with productivity $\pi^1 = \pi > 1$ (the leader), while all the other firms $j = 2, 3, \ldots$ have productivity $\pi^j = 1$ (the followers). $\pi \geq 1$ is the relative productivity of the highest type with respect to all the other firms. In such case, when there are $n_{i,t}$ active producers (one leader and $n_{i,t}-1$ followers) the price is equal to

$$p_{i,t} = \frac{1+(n_{i,t}-1)\,\pi}{n_{i,t}-(1-\rho)}\frac{\Theta_t}{\pi}$$

Therefore, the markup $\mu_{i,t}^{j} := \frac{p_{i,t}}{(\Theta_t/\pi^j)}$ charged by each type of firm is equal to

$$\begin{aligned}\mu_{i,t}^{L} &= \frac{1+(n_{i,t}-1)\,\pi}{n_{i,t}-(1-\rho)} \\ \mu_{i,t}^{F} &= \frac{(1/\pi)+(n_{i,t}-1)}{n_{i,t}-(1-\rho)}\end{aligned}$$

We can therefore see that, as π increases, the markup of the leader increases, while the markup of each follower decreases (when there is at least one follower producing, i.e, $n_{i,t} \geq 2$). We can also obtain an expression for the market share of each type of firm.

$$\begin{aligned}s_{i,t}^{L} &= \frac{1}{1-\rho}\frac{2-\pi-\rho+n_{i,t}(\pi-1)}{1+(n_{i,t}-1)\,\pi} \\ s_{i,t}^{F} &= \frac{1}{1-\rho}\frac{1-\rho\pi}{1+(n_{i,t}-1)\,\pi}\end{aligned}$$

The followers may produce provided $\rho\pi < 1$.[10] When there are $n_{i,t}$ firms, the profits that each follower makes are given by

$$\Pi^{F}(n_{i,t},\Theta_t,Y_t) := \begin{cases} 0 & \text{if } n_{i,t} \leq 1 \\ \left[\frac{1-\rho\pi}{1+(n_{i,t}-1)\,\pi}\right]^2\left[\frac{n_{i,t}-(1-\rho)}{1+(n_{i,t}-1)\,\pi}\pi\right]^{\frac{\rho}{1-\rho}}\frac{\Theta_t^{-\frac{\rho}{1-\rho}}Y_t}{1-\rho} & \text{if } n_{i,t} \geq 2 \end{cases} \tag{8}$$

As before, profits are increasing in aggregate demand/output Y_t. What is perhaps interesting to see in this example is how the profits of the followers change with the productivity of the leader, π. It can be shown that (8) is a decreasing function of π, the productivity of the leader. Therefore, the higher is the productivity of the leader, the lower the profits that each follower will make (*ceteris paribus*). This result is illustrated in Figure 5 below. It shows how the profits of a follower change with π, for fixed $n_{i,t} = 2$ (i.e. a duopoly with the leader and only one follower) and given aggregate output Y_t and factor cost index Θ_t.

Figure 5 also helps us understand the mechanism underlying the model. It illustrates the effects of a drop in aggregate output Y_t on the industry equilibrium. Suppose that aggregate output decreases from Y_0 to $Y_1 < Y_0$. If the leader has a small productivity advantage (for instance $\pi = \underline{\pi}$), the follower makes lower profits, but remains in operation (as the profits are still above c_f). However, if the leader has a large productivity advantage (for instance $\pi = \overline{\pi}$), the follower will be forced to leave the market following the fall in aggregate output (as the profits are now below c_f).

[10] Otherwise, the leader's monopoly price is below the followers' marginal cost

$$p_{i,t}(n_{i,t}=1) = \frac{1}{\rho}\frac{\Theta_t}{\pi} < \Theta_t$$

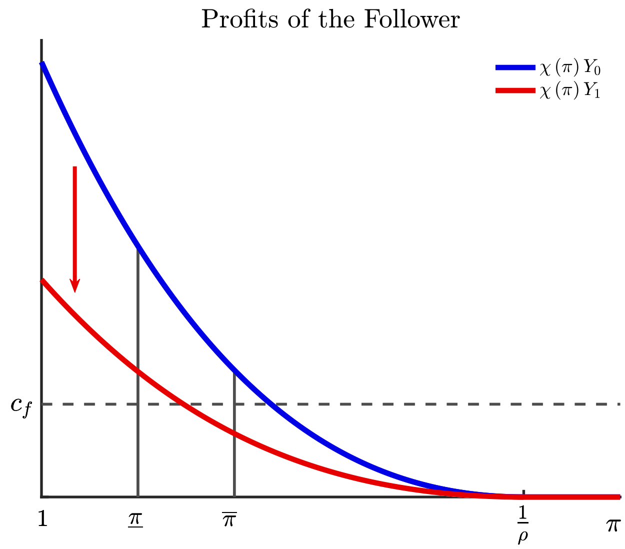


Figure 5: Profits of the follower

3.4 General Equilibrium

3.4.1 Aggregate Production Function

We now describe the general equilibrium of this economy. We will focus on a symmetric equilibrium in which all industries are identical and hence have n_t active firms. In such case, we can write aggregate output as a function of aggregate labor supply L_t and the aggregate capital stock K_t

$$Y_t = \Phi\left(\mathcal{F}_\pi, n_t\right)\; L_t^{1-\alpha} K_t^{\alpha}$$

The term $\Phi\left(\mathcal{F}_\pi, n_t\right)$ reflects aggregate TFP, for a productivity distribution $\mathcal{F}_\pi \coloneqq \left\{\pi^1, \pi^2, \pi^3, \ldots\right\}$ and when there are n_t firms per industry. As shown in Appendix (E.2), aggregate TFP can be written as

$$\Phi\left(\mu_t, \mathcal{F}_\pi, n_t\right) = \left(\sum_{k=1}^{n_t} \frac{s_t^k}{\pi_t^k}\right)^{-1}$$

where s_t^k is the market share of type π^k, given by equation (6). It can be shown that aggregate productivity decreases in the number of active firms n_t . The intuition is simple: as the number of active firms increases, less productive firms enter in the market and reduce the market share of the most productive units.

Lemma 1. *Aggregate productivity $\Phi\left(\mathcal{F}_\pi, n_t\right)$ decreases in the number of active firms, n_t.*

$$\Phi\left(\mathcal{F}_\pi, n_t + 1\right) < \Phi\left(\mathcal{F}_\pi, n_t\right)$$

Proof. See Appendix E.2. ■

3.4.2 Factor Prices and Factor Shares

To describe the law of motion of this economy, it is necessary to determine factor prices. We can write factor prices as

$$\begin{aligned} W_t = & \ (1-\alpha)\ \Theta(\mathcal{F}_\pi, n_t)\ L_t^{-\alpha} K_t^{\alpha} \\ R_t = & \ \alpha\, \Theta(\mathcal{F}_\pi, n_t)\ L_t^{1-\alpha} K_t^{\alpha-1} \end{aligned}$$

where $\Theta(\mathcal{F}_\pi, n_t)$ denotes the factor cost index in a symmetric equilibrium with n_t followers in every industry. The equilibrium factor cost index will be equal to

$$\Theta(\mathcal{F}_\pi, n_t) = \frac{n_t - (1-\rho)}{\sum_{k=1}^{n_t} \frac{1}{\pi^k}}$$

This function can be also shown to be increasing in the number of firms n_t, as stated in the following lemma.

Lemma 2. *The factor cost index $\Theta(\mathcal{F}_\pi, n_t)$ increases in the number of active firms, n_t*

$$\Theta(\mathcal{F}_\pi, n_t + 1) > \Theta(\mathcal{F}_\pi, n_t)$$

Proof. See Appendix E.3. ■

It therefore follows that, for any given aggregate labor supply L_t and capital stock K_t, the higher is the number of firms, the higher are factor prices W_t and R_t. The positive relationship between the aggregate number of firms (or the degree of product market competition) and factor prices is crucial to understand the model's mechanism, and in particular the existence of multiple equilibria and multiple steady-states. As more firms enter, competition becomes more intense, which induces a positive shift in factor demand and an increase in factor prices. The increase in factor prices will make agents increase their factor supply (larger labor supply and higher savings rate). As agents supply more factor of production, aggregate output increases - allowing the new entering firms to make positive profits.

We can also determine how the aggregate factor and profit shares vary with the number of active firms. Let $\Omega(\mathcal{F}_\pi, n_t)$ denote the aggregate factor share in a symmetric equilibrium where n_t firms produce in every industry, i.e.

$$\Omega(\mathcal{F}_\pi, n_t) := \frac{W_t\, L_t + R_t\, K_t}{Y_t}$$

Note that the aggregate profit share (exclusive of fixed production costs) is given by $1 - \Omega(\mathcal{F}_\pi, n_t)$. As shown in Appendix E.3, we have that

$$\Omega(\mathcal{F}_\pi, n_t) = \frac{\Theta(\mathcal{F}_\pi, n_t)}{\Phi(\mathcal{F}_\pi, n_t)}$$

It can be easily proved that $\Omega(\mathcal{F}_\pi, n_t)$ is a positive function of the number of followers n_t. This is an intuitive result - as the number of firms increases in every industry, competition becomes more intense, so that the profit share decreases and the factor share increases.

Lemma 3. *The aggregate factor share* $\Omega(\mathcal{F}_\pi, n_t)$ *increases in the number of active firms,* n_t

$$\Omega(\mathcal{F}_\pi, n_t + 1) > \Omega(\mathcal{F}_\pi, n_t)$$

Proof. See Appendix E.3. ■

The following lemma states an important result of the model: it characterizes how the productivity distribution $\mathcal{F}_\pi$ affects the aggregate factor share $\Omega(\cdot)$.

Lemma 4. *Suppose there are* n_t *active firm in every industry and let* π^j *be a productivity type such that* $\pi^j \geq \frac{1}{n_t}\sum_{k=1}^{n_t} \pi^k$. *Suppose that* π^j *increases to* $\tilde{\pi}^j > \pi^j$ *but all other types remain unchanged. Then, the new distribution* $\tilde{\mathcal{F}}_\pi$ *is such that*

$$\Omega\left(\tilde{\mathcal{F}}_\pi, n_t\right) < \Omega(\mathcal{F}_\pi, n_t)$$

Proof. See Appendix E.3. ■

To understand this lemma, pick some productivity type π^j and let it increase identically in all industries. Then, if π^j is already above the average $\frac{1}{n_t}\sum_{k=1}^{n_t} \pi^k$, the aggregate factor share will decrease. In other words, when the right tail of the productivity distribution increases, the aggregate factor share decreases (and the profit share increases). The intuition is simple. In every industry, high productivity firms are larger and charge higher markups. As large firms are able to increase their markups even further, average market power rises.

3.4.3 Law of Motion

Having determined factor prices, we can now determine the law of motion of our economy. The aggregate labor supply as a function of the number of active firms n_t and the aggregate capital stock K_t can be written as

$$L_t = \left[\frac{(1-\alpha)\,\Theta(\mathcal{F}_\pi, n_t)}{\eta}\right]^{\frac{1}{\nu+\alpha}} K_t^{\frac{\alpha}{\nu+\alpha}} \tag{9}$$

Similarly, we can write aggregate output as

$$Y_t = \Phi(\mathcal{F}_\pi, n_t)\left[\frac{(1-\alpha)\,\Theta(\mathcal{F}_\pi, n_t)}{\eta}\right]^{\frac{1-\alpha}{\nu+\alpha}} K_t^{\alpha\frac{1+\nu}{\nu+\alpha}} \tag{10}$$

and the equilibrium interest rate as

$$R_t = \left(\frac{1-\alpha}{\eta}\right)^{\frac{1-\alpha}{\nu+\alpha}} \alpha\ \left[\Theta(\mathcal{F}_\pi, n_t)\right]^{\frac{\nu+1}{\nu+\alpha}}\ K_t^{-\nu\frac{1-\alpha}{\nu+\alpha}} \tag{11}$$

Given our assumption that households cannot have negative assets, the demand for credit will be exclusively driven by the corporate side. Equilibrium in the credit market therefore requires that investment equals aggregate savings, i.e.

$$K_{t+1} = s_t Y_t \tag{12}$$

Recall that the economy's savings rate (which coincides with the fraction of savers) is equal to

$$s_t = (\beta_m R_{t+1})^{\gamma} \tag{13}$$

We can combine equations (11), (12) and (13) to write the aggregate capital stock at $t+1$ as a function of total output at time t

$$K_{t+1} = \iota_K \Gamma_K \left(\mathcal{F}_\pi, n_{t+1}\right) Y_t^{\varepsilon_K} \tag{14}$$

where ι_K, $\Gamma_K \left(\mathcal{F}_\pi, n_{t+1}\right)$ and ε_K are defined in Appendix E.4. Finally, we can obtain a law of motion for output by combining equations (10) (defined at $t+1$) and (14)

$$Y_{t+1} = \iota_Y \Gamma_Y \left(\mathcal{F}_\pi, n_{t+1}\right) Y_t^{\varepsilon_Y} \tag{15}$$

where ι_Y, $\Gamma_Y \left(\mathcal{F}_\pi, n_{t+1}\right)$ and ε_Y are defined in Appendix E.4. This map determines Y_{t+1} given Y_t (the state variable) and the number of active firms n_{t+1}. The term $\Gamma_Y \left(\mathcal{F}_\pi, n_{t+1}\right)$ can be shown to be increasing in the number of active firms n_{t+1} (see Appendix E.4).

We only need to determine under which conditions there can be n_{t+1} active firms per industry. As shown in Appendix E.1.4, given aggregate output Y_t and a factor cost index Θ_t, when there are n firms in any particular industry, type π^j firm makes production profits

$$\Pi\left(\pi^j, n, \mathcal{F}_\pi, \Theta_t, Y_t\right) = \mathit{\Lambda}\left(\mathcal{F}_\pi, \pi^j, n\right) \Theta_t^{-\frac{\rho}{1-\rho}} Y_t$$

where $\mathit{\Lambda}\left(\mathcal{F}_\pi, \pi^j, n\right)$ is a function that is increasing in π^j and decreasing in n. A symmetric equilibrium with n firms per industry is possible provided that (i) the profits of the marginal follower π^n are not lower than the fixed production cost, i.e. $\Pi\left(\pi^n, n, \mathcal{F}_\pi, \Theta_t, Y_t\right) \geq c_f$ and (ii) if an additional entrepreneur were to enter in one industry, he could make no gain, i.e. $\Pi\left(\pi^{n+1}, n+1, \mathcal{F}_\pi, \Theta_t, Y_t\right) \leq c_f$.

The next lemma provides the conditions for the existence of a symmetric equilibrium with n firms per industry.

$$Y_t = \Phi\left(\mathcal{F}_\pi, n_t\right) \left[\frac{(1-\alpha)\, \Theta\left(\mathcal{F}_\pi, n_t\right)}{\eta}\right]^{\frac{1-\alpha}{\nu+\alpha}} K_t^{\alpha \frac{1+\nu}{\nu+\alpha}}$$

Lemma 5. *Suppose there is a symmetric equilibrium with n firms per industry at time $t+1$. Then Y_{t+1} is such that*

$$\underline{Y}(\mathcal{F}_\pi, n) \leq Y_{t+1} \leq \overline{Y}(\mathcal{F}_\pi, n)$$

where

$$\begin{aligned}\underline{Y}(\mathcal{F}_\pi, n) &:= \frac{c_f}{\Lambda(\mathcal{F}_\pi, n)}\left[\Theta(\mathcal{F}_\pi, n)\right]^{\frac{\rho}{1-\rho}} \\ \overline{Y}(\mathcal{F}_\pi, n) &:= \frac{c_f}{\Lambda(\mathcal{F}_\pi, n+1)}\left[\Theta(\mathcal{F}_\pi, n)\right]^{\frac{\rho}{1-\rho}}\end{aligned}$$

For such level of output to be achieved at $t+1$, Y_t must be such that

$$\underline{Y}_0(\mathcal{F}_\pi, n) \leq Y_t \leq \overline{Y}_0(\mathcal{F}_\pi, n)$$

where

$$\begin{aligned}\underline{Y}_0(\mathcal{F}_\pi, n) &:= \left\{\iota_Y^{-1}\frac{c_f}{\Lambda(\mathcal{F}_\pi, n)}\left[\Phi(\mathcal{F}_\pi, n)\right]^{-1}\left[\Theta(\mathcal{F}_\pi, n)\right]^{\frac{\rho}{1-\rho}-\varsigma_Y}\right\}^{\frac{1}{\varepsilon_Y}} \\ \overline{Y}_0(\mathcal{F}_\pi, n) &:= \left\{\iota_Y^{-1}\frac{c_f}{\Lambda(\mathcal{F}_\pi, n+1)}\left[\Phi(\mathcal{F}_\pi, n)\right]^{-1}\left[\Theta(\mathcal{F}_\pi, n)\right]^{\frac{\rho}{1-\rho}-\varsigma_Y}\right\}^{\frac{1}{\varepsilon_Y}}\end{aligned}$$

Proof. See Appendix E.5. ■

The economy can also exhibit asymmetric equilibria. This happens for instance when

$$\overline{Y}_0(\mathcal{F}_\pi, n) < \underline{Y}_0(\mathcal{F}_\pi, n+1)$$

i.e. the minimum output level at time t that can sustain a symmetric equilibrium with $n+1$ firms per industries at $t+1$ is larger than the maximum output level at time t that can sustain a symmetric equilibrium with n firms per industries at $t+1$. In such case, when

$$\overline{Y}_0(\mathcal{F}_\pi, n) < Y_t < \underline{Y}_0(\mathcal{F}_\pi, n+1)$$

there will be an asymmetric equilibrium at time $t+1$: some industries will contain n firms, whereas some industries will contain $n+1$ firms. The equilibrium conditions are described in Appendix E.6.

Lemma 5 states the conditions under which a symmetric equilibrium with n firms is possible. For example, whenever $\underline{Y}_0(\mathcal{F}_\pi, n) \leq Y_t \leq \overline{Y}_0(\mathcal{F}_\pi, n+1)$, the economy can sustain an equilibrium in which n firms produce in every industry at time $t+1$. Note however that Lemma 5 does make a statement about uniqueness. Indeed, suppose that $\underline{Y}_0(\mathcal{F}_\pi, n+1) < \overline{Y}_0(\mathcal{F}_\pi, n)$. In such case, the *minimum* level of time t output that allows for a symmetric equilibrium with $n+1$ firms at time $t+1$ is lower that the *maximum* level of time t output that is consistent with n firms at time $t+1$. If we further have $\underline{Y}_0(\mathcal{F}_\pi, n+1) \leq Y_t \leq \overline{Y}_0(\mathcal{F}_\pi, n)$, then at least two equilibria are possible at time $t+1$: a symmetric equilibria with n firms per industry, and a symmetric equilibria with $n+1$ firms.[11] The next lemma states the conditions under which

[11] As we shall see, a third equilibrium may also be possible: an asymmetric equilibrium in which some industries have n firms, while some

multiple equilibria can arise.

Lemma 6. *(Existence of Multiple Equilibria) Suppose that* $Y_t \in \left[\underline{Y}_0(\mathcal{F}_\pi, n), \overline{Y}_0(\mathcal{F}_\pi, n)\right]$*, so that a symmetric equilibrium with n firms is possible at* $t+1$*. If*

$$\frac{\Phi(\mathcal{F}_\pi, n)}{\Phi(\mathcal{F}_\pi, n+1)} < \left[\frac{\Theta(\mathcal{F}_\pi, n)}{\Theta(\mathcal{F}_\pi, n+1)}\right]^{\frac{\rho}{1-\rho}-\varsigma_Y}$$

then there exists an interval $I_{n+1} \subset \left[\underline{Y}_0(\mathcal{F}_\pi, n), \overline{Y}_0(\mathcal{F}_\pi, n)\right]$ *in which a symmetric equilibrium with* $n+1$ *firms is also possible at* $t+1$*.*

Proof. See Appendix E.3. ■

Corollary. *Suppose all firms are equally productive. In such case, there is equilibrium multiplicity if*

$$\rho < \frac{(1+\gamma) + \alpha(\nu - 1)}{(1+\gamma)(1+\nu)}$$

The following lemma describes how a change in the productivity distribution $\mathcal{F}_\pi$ can affect the thresholds $\underline{Y}(\mathcal{F}_\pi, n)$ and $\underline{Y}_0(\mathcal{F}_\pi, n)$.

Lemma 7. *Let* $\mathcal{F}_\pi := \left\{\pi^1, \pi^2, \pi^3, \ldots\right\}$ *denote the distribution of productivity types in any given industry. We have that for any type* $\pi^k > \pi^n$

$$\frac{\partial \underline{Y}(\mathcal{F}_\pi, n)}{\partial \pi^k} > 0$$

Moreover, we also have that

$$\frac{\partial}{\partial \pi^k}\left[\frac{\underline{Y}_0(\mathcal{F}_\pi, n)}{\underline{Y}(\mathcal{F}_\pi, n)}\right] > 0$$
$$\Leftrightarrow \quad -\left[\Phi(\mathcal{F}_\pi, n)\right]^{-1}\frac{\partial \Phi(\mathcal{F}_\pi, n)}{\partial \pi^k} - \varsigma_Y\left[\Theta(\mathcal{F}_\pi, n)\right]^{-1}\frac{\partial \Theta(\mathcal{F}_\pi, n)}{\partial \pi^k} + (1-\varepsilon_Y)\left[\underline{Y}(\mathcal{F}_\pi, n)\right]^{-1}\frac{\partial \underline{Y}(\mathcal{F}_\pi, n)}{\partial \pi^k} > 0$$

Proof. See Appendix E.5. ■

This lemma states an important result of the model. Suppose that π^1 increases. Then the minimum level of output that is consistent the entry of type π^2 in all industries increases, i.e. it becomes more difficult for types π^2 to profitably enter.

To understand how the model works, let us consider two examples.

No productivity differences Let us start by assuming that all firms are equally productive, so that $\pi^j = 1$. In such case, aggregate TFP is constant and equal to one (thus independent of the number of firms).

$$\Phi(\mathcal{F}_\pi, n_t) = 1$$

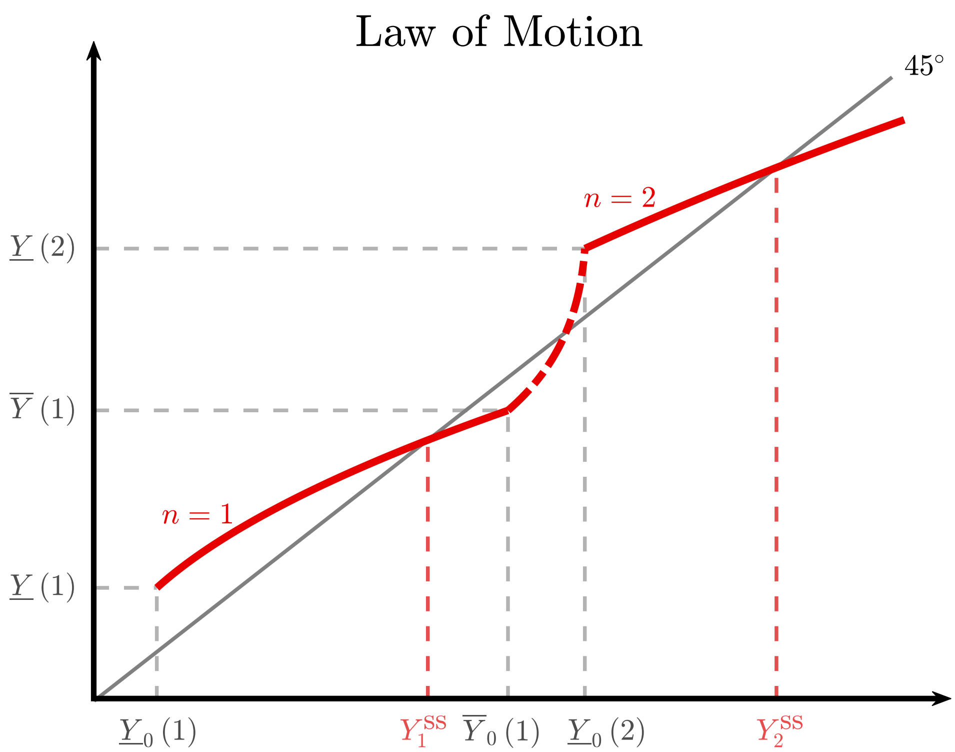
---

others have $n+1$ firms.

The number of firms will affect the aggregate factor. We have that

$$\Omega\left(\mathcal{F}_\pi, n_t\right) = \frac{n_t - (1-\rho)}{n_t}$$

The aggregate factor share thus goes from $\Omega\left(n_t = 0\right) = \rho$ (monopoly) to $\Omega\left(n_t \to \infty\right) = 1$ (perfect competition). Given that aggregate productivity is constant and equal to one, the factor cost index coincides with the aggregate factor share, $\Theta\left(\mathcal{F}_\pi, n_t\right) = \Omega\left(\mathcal{F}_\pi, n_t\right)$. Figure 6 below shows the law of motion of this economy for a particular value of aggregate productivity μ_t: it represents the map defined by equation (15) and Lemma 5. The full lines show the region in which the economy exhibits a symmetric equilibrium (i.e. all industries have the same number of firms). The dashed line represents the asymmetric equilibrium described in Appendix E.6 (some industries are a monopoly, while others consist of a duopoly).



Parameters: $\rho = 0.65$, $\alpha = 0.4$, $\nu = 0.5$, $\eta = 0.2$, $\gamma = 1$ and $c_f = 0.075$

Figure 6: Law of Motion: No productivity differences

There are two steady-states with a symmetric equilibrium: one where all industries are a monopoly (Y_1^{ss}), and another one where all industries are a duopoly (Y_2^{ss}). Note that despite the existence of two steady-states, there is a unique equilibrium: given the particular realization of μ_t, there is a unique value of Y_{t+1} for each value of Y_t (the state variable).[12]

[12] Figure 24 in Appendix 6 shows an example of a law of motion that features multiple equilibria.

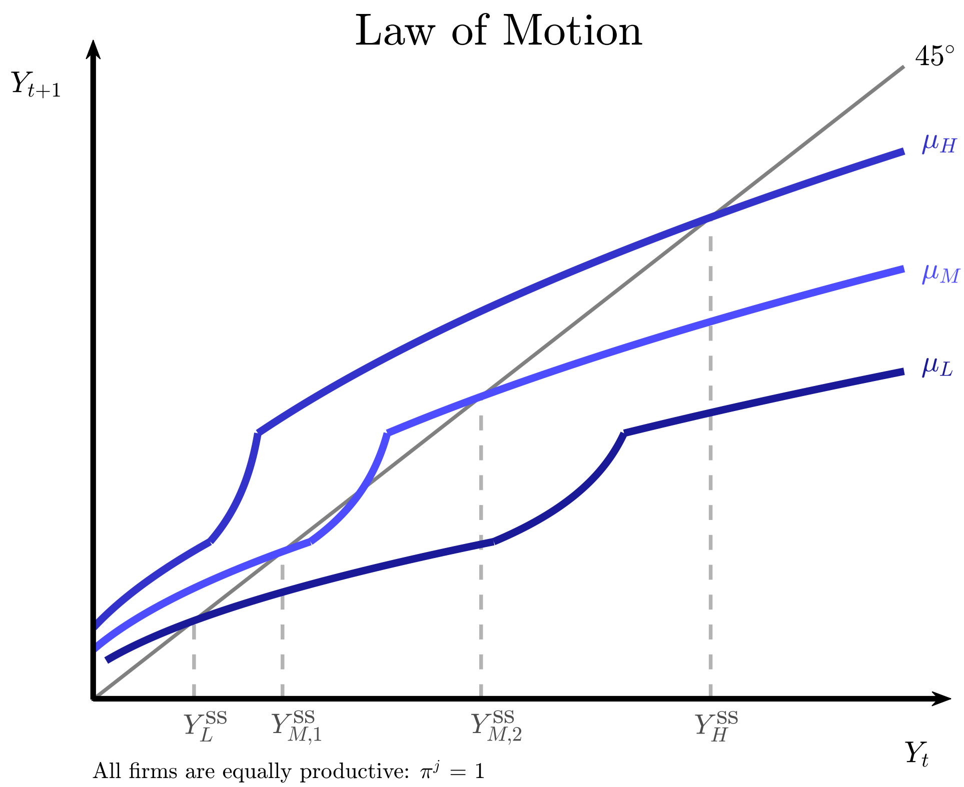


Figure 7: Law of Motion: No productivity differences

Let us now analyze the consequences of aggregate fluctuations. For simplicity, suppose that aggregate productivity can only take three values: a low value μ_L, an intermediate value μ_M and a high value μ_H. Figure 7 represents the law of motion under each value of aggregate productivity. As one can see, when aggregate productivity is low and equal to μ_L, the economy exhibits a unique steady-state where all industries are a monopoly (Y_L^{ss}). Under μ_H, on the other hand, there is only a unique steady-state where all industries are a duopoly (Y_H^{ss}). Finally, when aggregate productivity takes the intermediate value μ_M, the economy exhibits two steady-states: a low one where all industries are a monopoly ($Y_{M,1}^{ss}$) and a high one where all industries are a duopoly ($Y_{M,2}^{ss}$).

To exemplify the dynamics of the model, suppose that aggregate productivity is equal to μ_H and that the economy is at the steady-state Y_H^{ss}. Suppose now that there is a negative aggregate productivity shock. If aggregate productivity falls to μ_M and remains there for a long period, the economy will converge to the new steady-state $Y_{M,2}^{ss}$. Output is lower than before, but the market structure is identical - all sectors are still a duopoly. The transition dynamics is represented in Figure 8 below.

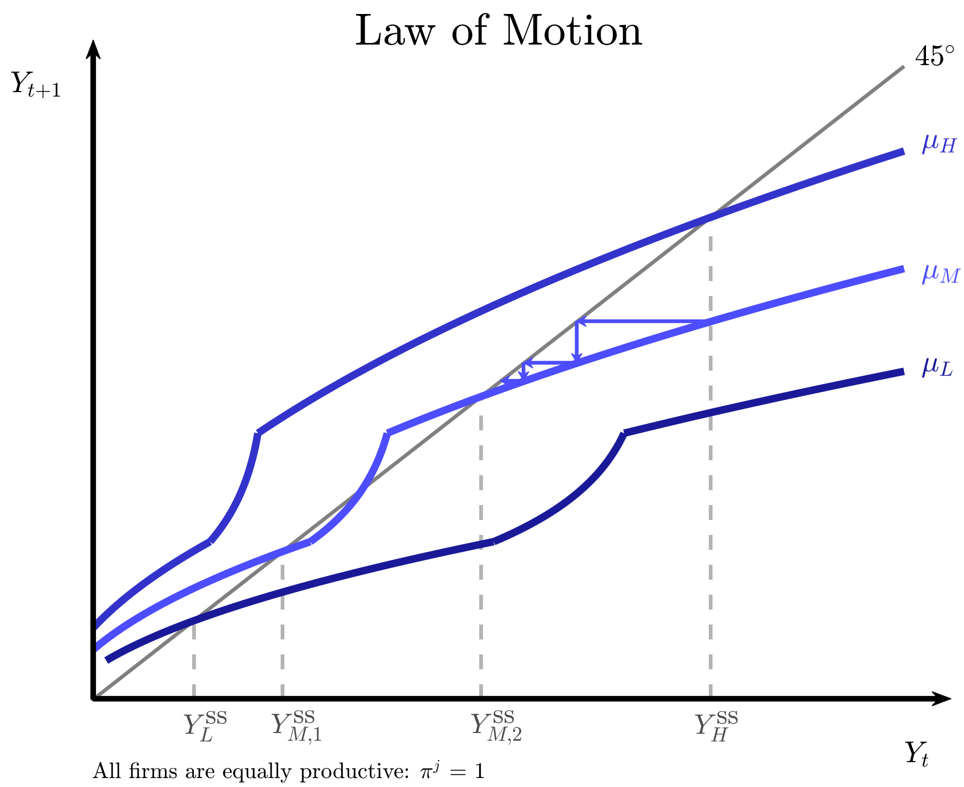


Figure 8: Law of Motion: No productivity differences

Now suppose that aggregate productivity falls to μ_L and remains there for a long period. In such case, the economy will converge to the new steady-state Y_L^{ss}. Note now that in the new steady-state all sectors are a monopoly. The economy therefore moves to a more competitive market structure. The transition dynamics is represented in Figure 9 below.

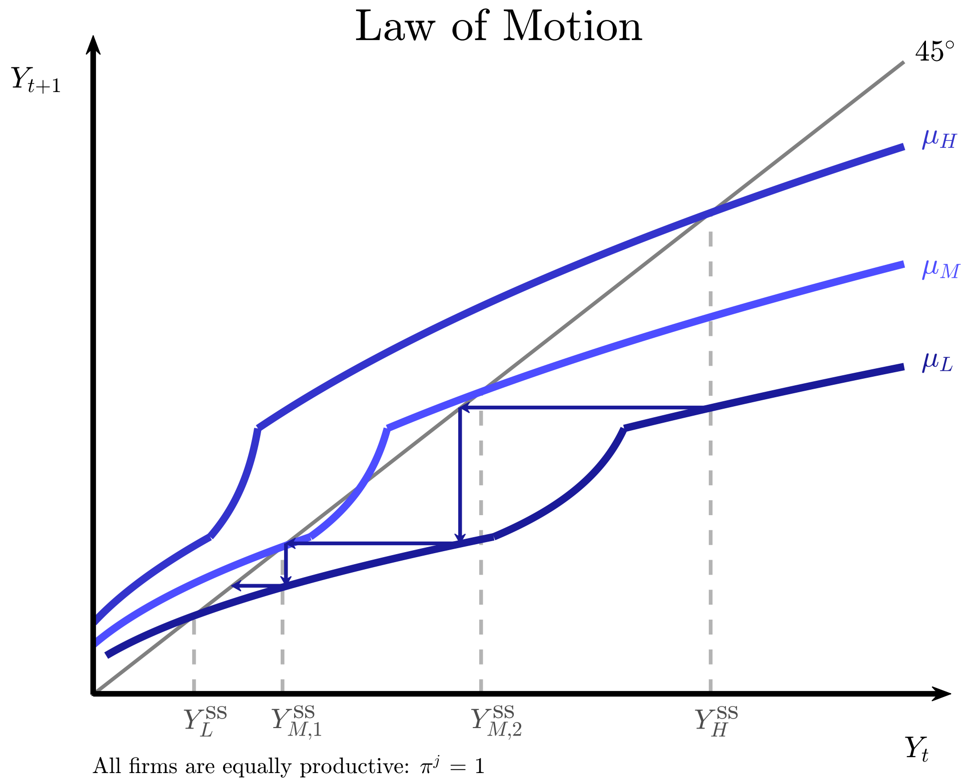


Figure 9: Law of Motion: No productivity differences

Figure 10 below shows the evolution of the number of firms per industry and the aggregate profit share. Figure 11 shows the evolution of the investment and interest rates, as well as labor market participation (which in our model coincides with employment) and the labor share. As we can see, we can replicate the patterns of Figures 3 and 2.

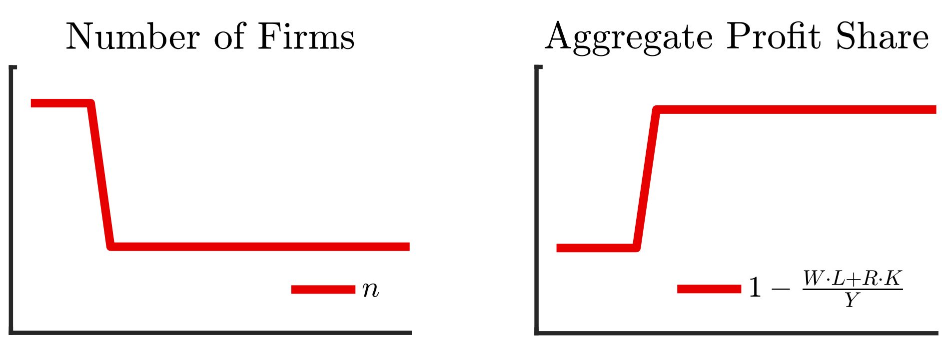


Figure 10: Transition to new steady-state

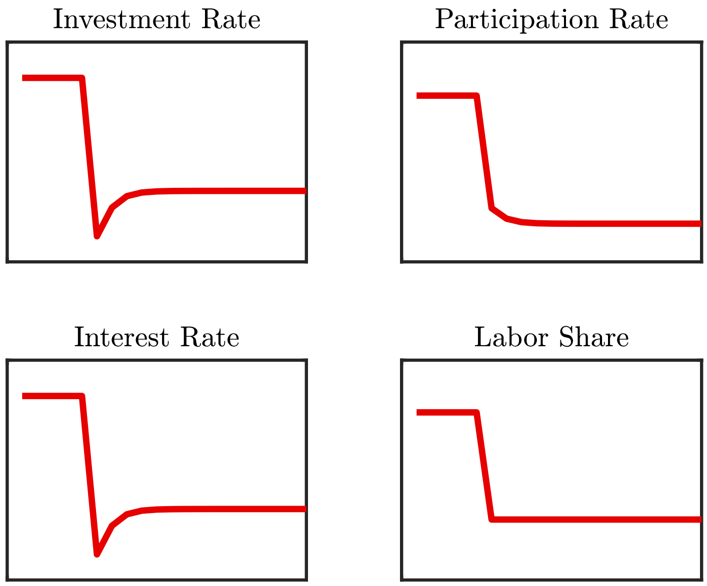


Figure 11: Transition to new steady-state

We now let aggregate productivity fluctuate according to equation (4). Figure 12 below shows the distribution of output. There are two stochastic steady-states: a low one where all sectors are a monopoly, and a high one where all sectors are a duopoly. Given the particular parameters chosen, the economy is mostly around the high steady-state.

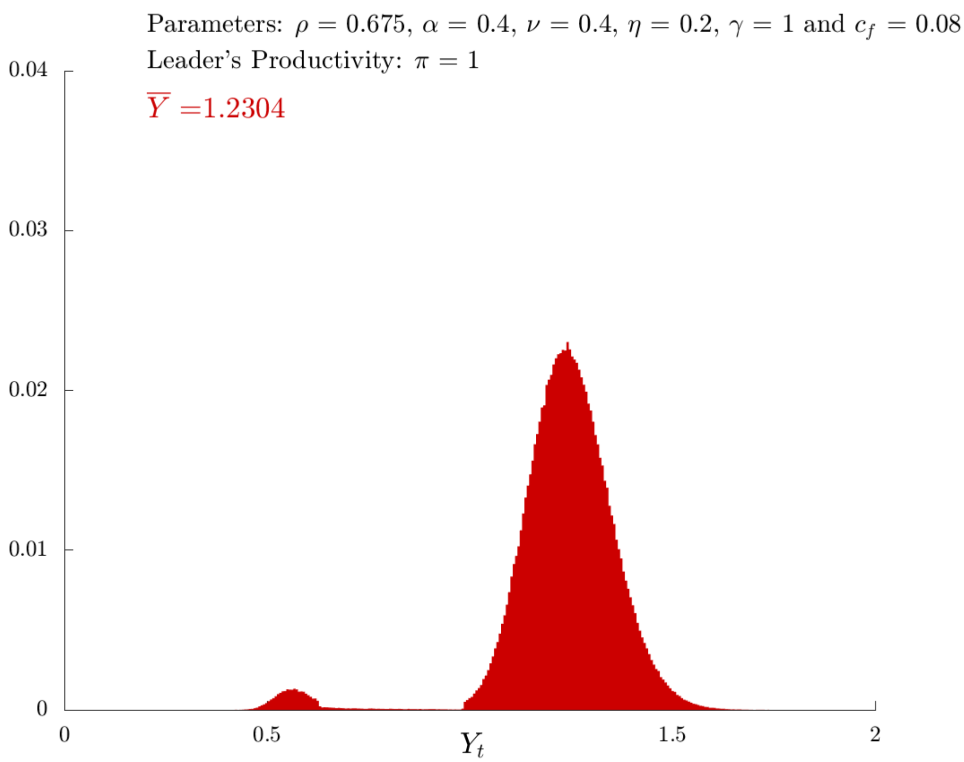


Figure 12: Output Distribution: No productivity differences

One leader *versus* multiple followers Let us introduce a modification in the previous example. Suppose that now there is a productive firm with productivity $\pi^L = \pi > 1$ (the leader), while all the other firms $j = 2, 3, \ldots$ have productivity $\pi^F = 1$ (the followers). All industries will consist of a monopoly (where the leader produces) provided that

$$\frac{c_f}{1-\rho} \leq Y_t \leq (1-\rho)\, c_f \left(\frac{1+\pi}{1-\rho\pi}\right)^2 \left(\frac{1+\pi}{1+\rho}\rho\right)^{\frac{\rho}{1-\rho}}$$

A duopoly will however be possible in all industries provided that

$$Y_t \geq (1-\rho)\, c_f \left(\frac{1+\pi}{1-\rho\pi}\right)^2$$

As we can see from the above condition, the higher is the relative productivity of the leader (π), the higher is the threshold required for a duopoly to start. Let us analyze what happens in this example after an increase π.

Figure 13 represents the effects of a small increase in π. The law of motion under the initial value of π is represented in light red. The law of motion under the increase in π is represented in blue. As we can see, after the increase in the leaders' productivity π, the curve representing a symmetric equilibrium with $n = 2$ firms per industry moves up, i.e. $\uparrow \underline{Y}(\mathcal{F}_\pi, 2)$, and to the right, i.e. $\uparrow \underline{Y}_0(\mathcal{F}_\pi, 2)$.

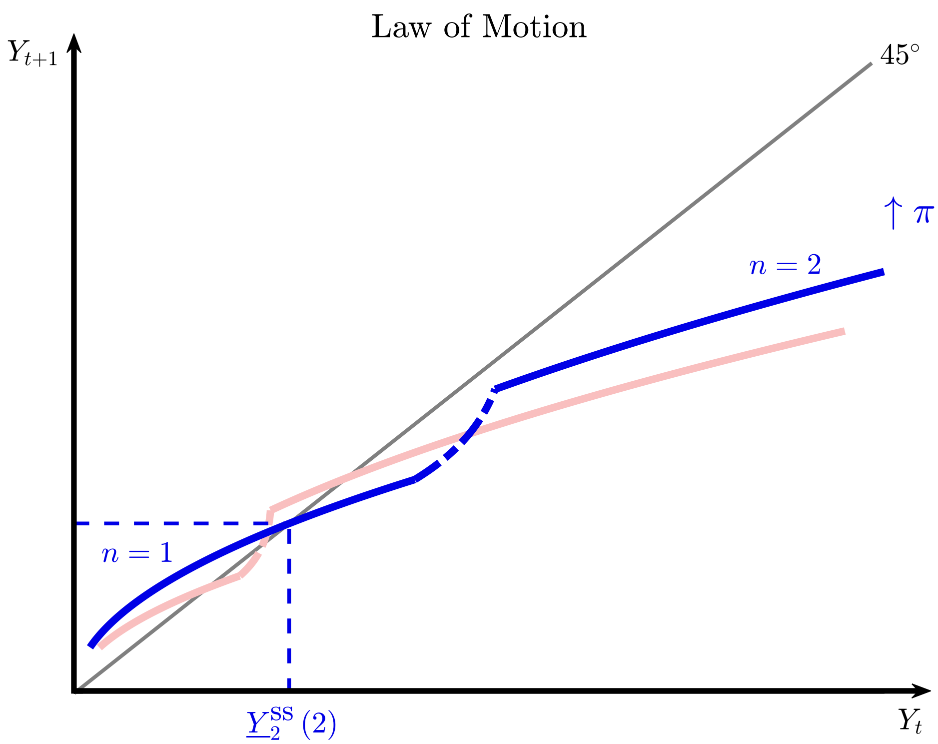


Figure 13: Law of Motion: Leader *versus* Followers

Figure 14 below replicates the example with three values for aggregate productivity. The main difference with respect to Figure 7 is that now, for the intermediate value of aggregate productivity μ_M. Note that the two steady-states Y_L^{ss} and Y_H^{ss} are larger after the increase in the leaders' productivity. The same happens with the low steady-state when aggregate productivity is equal to μ_M ($Y^{ss}_{M,1}$). This result is not surprising. Keeping the market structure constant (for example, a monopoly in every industry), the higher is the productivity of the leader, the higher is aggregate output. Figure 14 shows however that it becomes increasingly more difficult to sustain a duopoly and, as a consequence, the steady-state $Y^{ss}_{M,2}$ disappears.

Note that now, if the economy starts at the high steady-state Y^{ss}_H and aggregate productivity falls to μ_M, the economy will converge to $Y^{ss}_{M,1}$, where all industries are a monopoly.

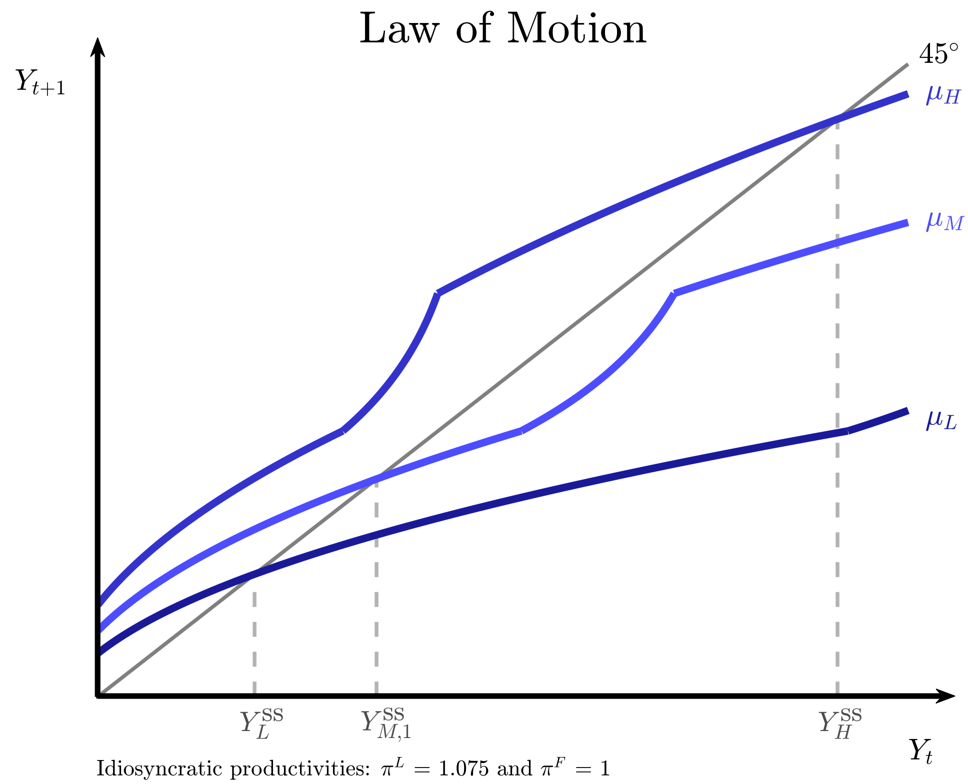


Figure 14: Law of Motion: Leader *versus* Followers

Finally, we let again aggregate productivity fluctuate according to equation (4). Figure 15 below shows the distribution of output. There are again two steady-states, which are higher than the ones before the increase in π. However, now the economy is more likely to be around the low steady-state. As a consequence, average output decreases.

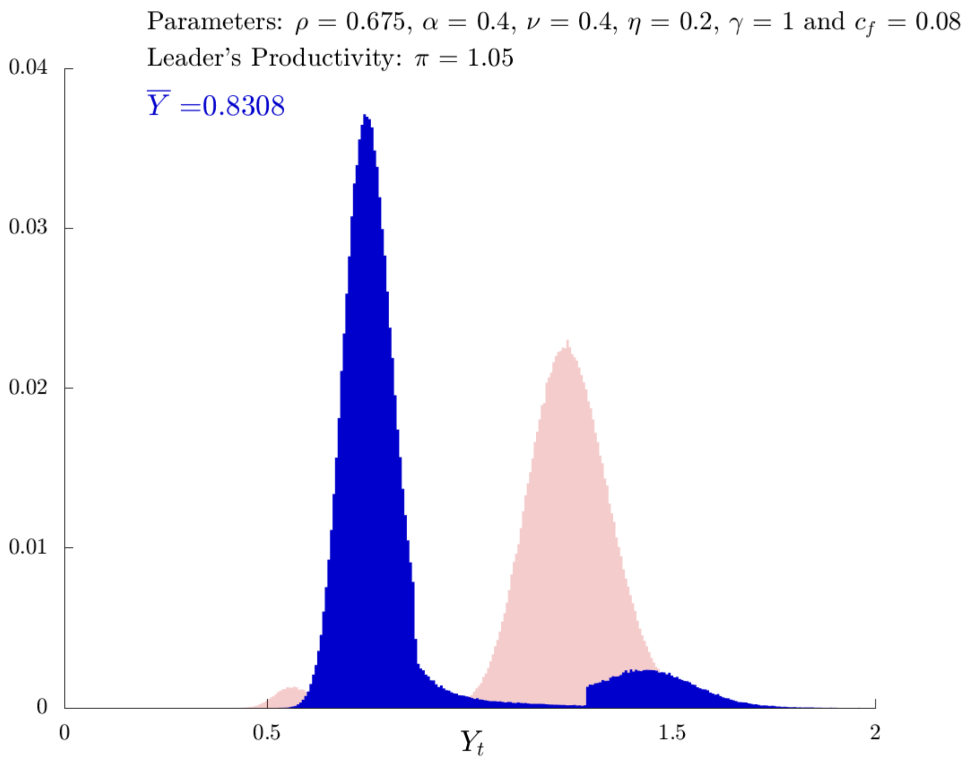


Figure 15: Output Distribution: Leader *versus* Followers

4 Quantitative and Policy Evaluation

4.1 Preferences

There is a representative household with lifetime utility

$$U_{i,t} = \mathbb{E} \sum_{t=0}^{\infty} \beta^t \, U\left(C_t, L_t\right)$$

The period utility function is

$$U\left(C_{i,t}, L_{i,t}\right) = \frac{1}{1-\delta} \left(c_{i,t} - \frac{l_{i,t}^{1+\nu}}{1+\nu} \right)^{1-\delta}$$

where $0 \leq \delta \leq 1$ and $\nu > 0$. Denoting by $a_{i,t}$ the assets of individual i at time t and by $T_{i,t}$ any lump sum transfer he receives (for example, the profits generated in any firm he has created), we have the following dynamic budget constraint

$$K_{t+1} = \left[r_t + (1-\delta)\right] K_t + W_t L_t + \Pi_t - C_t$$

4.2 Technology

The final good is a CES aggregate of I different varieties

$$Y_t = \left(\sum_{i=1}^{I} y_{i,t}^{\rho} \right)^{\frac{1}{\rho}}$$

where $y_{i,t}$ is the quantity of variety $i \in [0,1]$ and $\sigma_I \equiv \dfrac{1}{1-\rho}$ is the elasticity of substitution across industries. The output of each industry i is itself a CES composite of differentiated goods

$$y_{i,t} = \left(\sum_{j=1}^{n_{i,t}} y_{ji,t}^{\eta} \right)^{\frac{1}{\eta}}$$

where $n_{i,t}$ is the number of active firms in industry i at time t, and $\sigma_G \equiv \dfrac{1}{1-\eta}$ is the within-industry elasticity of substitution. Following Atkeson and Burstein (2008), we assume that the within-industry elasticity of substitution is larger than the across-industry elasticity of substitution.

Assumption. $0 < \rho < \eta < 1$

4.3 Calibration

Parameter Values

Description	Parameter	Value	Source/Target
Across-Industry ES	σ_I	1	Atkeson and Burstein (2008)
Within-Industry ES	σ_G	10	Atkeson and Burstein (2008)
Capital Elasticity	α	1/3	Standard Value
Elasticity of Labor Supply	ν	0.4	Jaimovich and Rebelo (2009)
Discount Factor	β	0.96	Standard Value
Depreciation Rate	δ	0.05	Standard Value
Number of Industries	I	200	See text
Maximum Number of Firms per Industry	N	15	See text
Pareto Tail	λ	0.4	
Maximum Fixed Cost	$\bar{c}$	0.01	

5 Conclusion

We have developed a general equilibrium, multi-industry model with oligopolistic competition and variable markups. Our model features a positive interaction between aggregate demand and product market competition that may give rise to multiple steady-states. In particular there can be high steady-states in which intense competition, by generating high factor prices, sustains high factor supply and high aggregate demand/output; and low steady-states where weak competition depresses factor prices and, through this channel, factor supply and factor demand. We refer to such low steady-states as *low competition traps*. An important contribution of our theory is to show that the likelihood that the economy falls in a *low competition trap* increases with the degree of markup dispersion.

Our model can rationalize some of the facts discussed in Section 1. In particular, it explains how a long-run increase in the variance of the firm markup distribution - as documented by De Loecker, Eeckhout and Unger (2018) - may have contributed to the severity of the 2008 recession and resulted in a persistent deviation of per capita GDP from trend. In addition to this fact, our model can rationalize a number of post-2008 trends, such as the joint decline of investment and interest rates, as well as the persistent decline in labor market participation and the significant drop of the labor share.

We are currently working on an extended version of our model that permits a serious quantitative evaluation of the mechanism that we propose. Among other things, we want to quantitatively assess the importance of rising markup dispersion in the amplification and propagation of the 2008 crisis. We also plan to conduct a number of policy experiments (such as the introduction of firm entry subsidies) and perform a welfare analysis. As mentioned earlier, subsidizing firm entry will have ambiguous welfare-implications in our model. The entry of an unproductive entrant entails an obvious cost (the fixed production cost and the factors it spends on production), but also brings a benefit (as it adds competitive pressure on the market, forcing the productive incumbents to expand).

We also plan to introduce endogenous growth in a future extension. As documented in Figure 1, real GDP per capita exhibits a widening deviation from trend, which indicates that growth rates have been lower after 2008. We think that an extended version of our model with endogenous R&D can potentially account for this fact. In a world where firms conduct R&D because of an *escape-from-competition* effect, a decrease in product market competition will likely reduce firms' incentives to innovate.

References

[1] Andrews, D., C. Criscuolo and P. Gal (2015), "Frontier Firms, Technology Diffusion and Public Policy," The Future of Productivity: Main Background Papers, OECD.

[2] Anzoategui, D., D. Comin, M. Gertler and J. Martinez (2016), "Endogenous Technology Adoption and R&D as Sources of Business Cycle Persistence," NBER Working Paper 22005.

[3] Atkeson, A. and A. Burstein (2008), "Pricing to Market, Trade Costs, and International Relative Prices," *American Economic Review* 98(5), 1998-2031.

[4] Autor, D., D. Dorn, L. Katz, C. Patterson and J. Van Reenen (2017), "The Fall of the Labor Share and the Rise of Superstar Firms", NBER Working Paper 23396.

[5] Barkai, S. (2017), "Declining Labor and Capital Shares", Working Paper.

[6] Benigno, G. and L. Fornaro (2018), "Stagnation Traps," *Review of Economic Studies* 85(3), 1425-1470.

[7] Bessen, J. (2016), "Accounting for Rising Corporate Profits: Intangibles or Regulatory Rents?," Boston University School of Law, Law & Economics Paper No 16-18.

[8] Brinca, P., V. Chari, P. Kehoe and E. McGrattan (2016), "Accounting for Business Cycles," In: Taylor, J. and H. Uhlig (Eds.), Handbook of Macroeconomics Vol. 2A, 1013-1063.

[9] Chatterjee, S., R. Cooper and B. Ravikumar (1993), "Strategic Complementarity in Business Formation: Aggregate Fluctuations and Sunspot Equilibria," *Review of Economic Studies* 60(4), 795-811.

[10] Decker, R., J. Haltiwanger, R. Jarmin and J. Miranda (2014), "The Secular Decline in Business Dynamism in the U.S.," Working Paper.

[11] Decker, R., J. Haltiwanger, R. Jarmin and J. Miranda (2017), "Changing Business Dynamism and Productivity: Shocks vs. Responsiveness," Working Paper.

[12] De Loecker, J. and J. Eeckhout (2018), "Global Marker Power," Working Paper.

[13] De Loecker, J., J. Eeckhout and G. Unger (2018), "The Rise of Market Power and the Macroeconomic Implications," Working Paper.

[14] Edmond, C., V. Midrigan and D. Xu (2018), "How Costly Are Markups?," Working Paper.

[15] Eggertsson, G. J. Robbins and E. Wold (2018), "Kaldor and Piketty's Facts: The Rise of Monopoly Power in the United States," NBER Working Paper 24287.

[16] Elsby, M., B. Hobijn and A. Şahin (2013), "The Decline of the U.S. Labor Share," Brookings Papers on Economic Activity 44(2), 1-63.

[17] Fernald, J., R. Hall, J. Stock and M. Watson (2017), "The Disappointing Recovery of Output after 2009," Brookings Papers on Economic Activity 48(1), 1-81.

[18] Gilchrist, S., R. Schoenle, J. Sim and E. Zakrajšek (2017), "Inflation Dynamics during the Financial Crisis," *American Economic Review* 107(3), 785-823.

[19] Guerron-Quintana, P. and R. Jinnai (2018), "Financial Frictions, Trends, and the Great Recession," *Quantitative Economics*, forthcoming.

[20] Gutiérrez, G. and T. Philippon (2017), "Declining Competition and Investment in the US," NBER Working Paper 23583.

[21] Hall, R. (2018), "Using Empirical Marginal Cost to Measure Market Power in the US Economy," Working Paper.

[22] Jones, C. and T. Philippon (2017), "The Secular Stagnation of Investment", Working Paper.

[23] Karabarbounis, L. and B. Neiman (2013), "The Global Decline of the Labor Share," *Quarterly Journal of Economics* 129(1), 61-103.

[24] Karabarbounis, L. and B. Neiman (2018), "Accounting for Factorless Income," NBER Working Paper 24404.

[25] Kehrig, M. (2015), "The Cyclical Nature of the Productivity Distribution," Working Paper.

[26] Kehrig, M. and N. Vincent (2018), "The Micro-Level Anatomy of the Labor Share Decline," NBER Working Paper 25275.

[27] Liu, E. A. Mian and A. Sufi (2019), "Low Interest Rates, Market Power, and Productivity Growth," Working Paper.

[28] Rotemberg, J. and Saloner, G. (1986), "A Supergame-Theoretic Model of Price Wars during Booms," *American Economic Review* 76(3), 390-407.

[29] Schaal, E. and M. Taschereau-Dumouchel (2015), "Coordinating Business Cycles", Working Paper.

[30] Song, J., D. Price, F. Guvenen, N. Bloom and T. von Wachter (2019), "Firming Up Inequality," *Quarterly Journal of Economics* 134(1), 1-50.

[31] Syverson, C. (2019), "Macroeconomics and Market Power: Facts, Potential Explanations and Open Questions," *The Brookings Economic Studies.*

[32] Traina, J. (2018), "Is Aggregate Market Power Increasing? Production Trends Using Financial Statements," Stigler Center New Working Paper Series No 17.

[33] Van Reenen, J. (2018), "Increasing Differences Between Firms: Market Power and the Macro-Economy," CEP Discussion Paper 1576.

A US Real GDP per capita

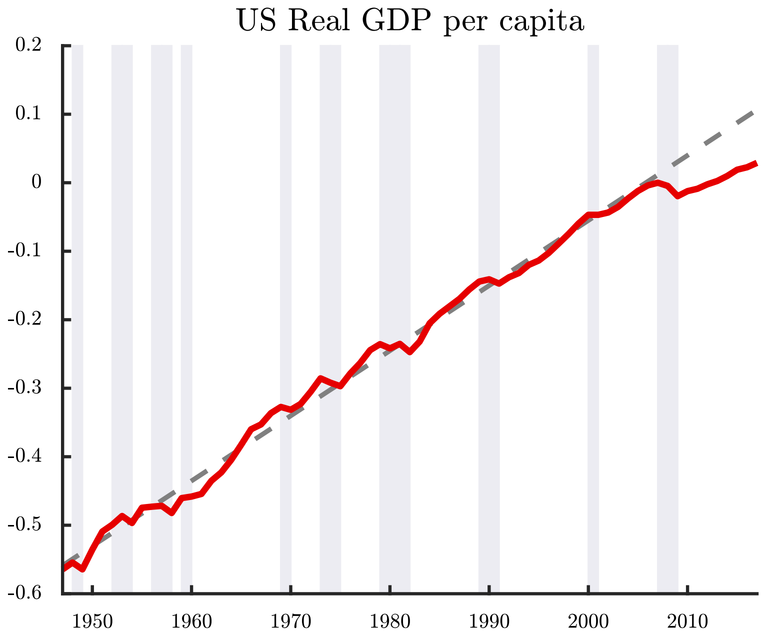


Figure 16: **US Real GDP per capita: 1947-2017**
Data is from the US Bureau of Economic Analysis. The series shown is on logs, undetrended and centered around 2007. The linear trend is computed for the 1947-2007 period.

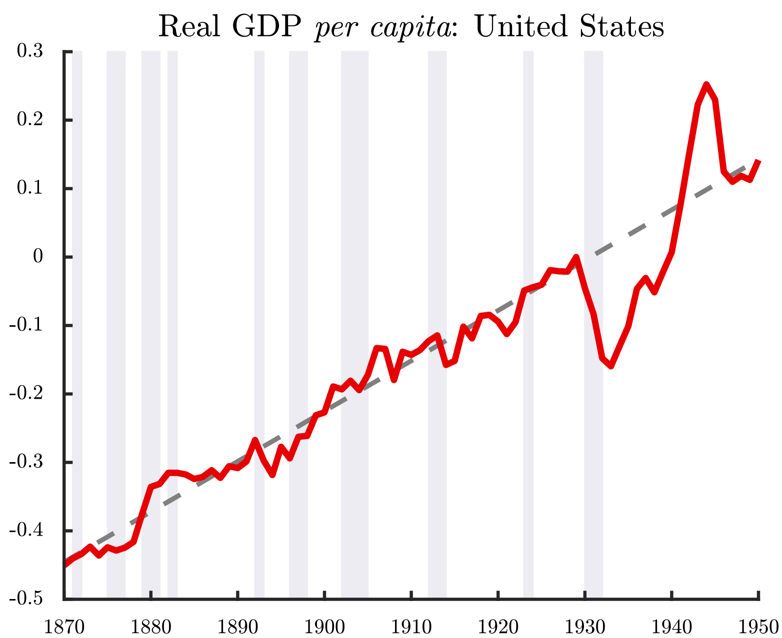


Figure 17: **US Real GDP per capita: 1870-1950**
Data is from Angus Maddison, "Historical Statistics of the World Economy: 1-2008 AD", University of Groningen. The series shown is on logs, undetrended and centered around 1929. The linear trend is computed for the 1870-1929 period.

B Labor Share

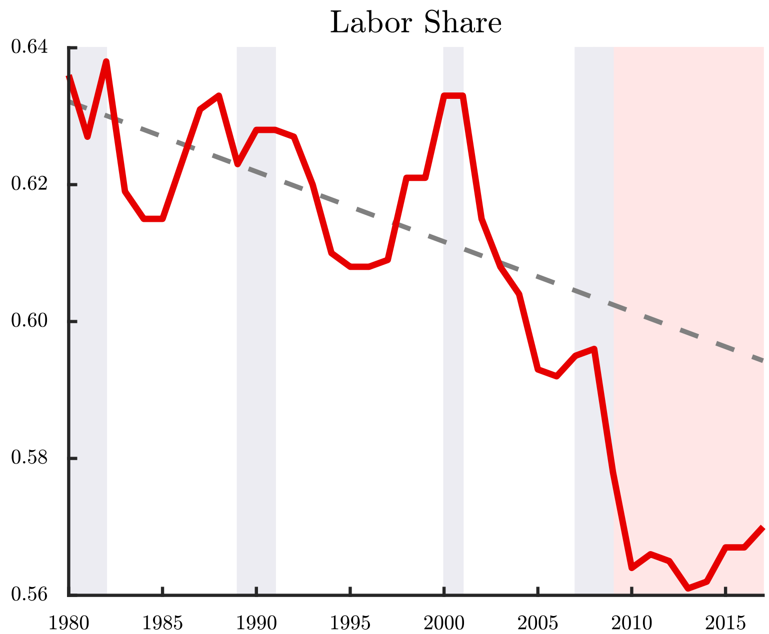


Figure 18: **US Labor Share: 1980-2017**
The labor share is ratio of total worker compensation to nominal output, for the US nonfarm business sector (from the BLS). The dashed grey line shows a linear trend computed for the 1980-2007 period.

C Number of Firms

C.1 Number of Firms per Sector

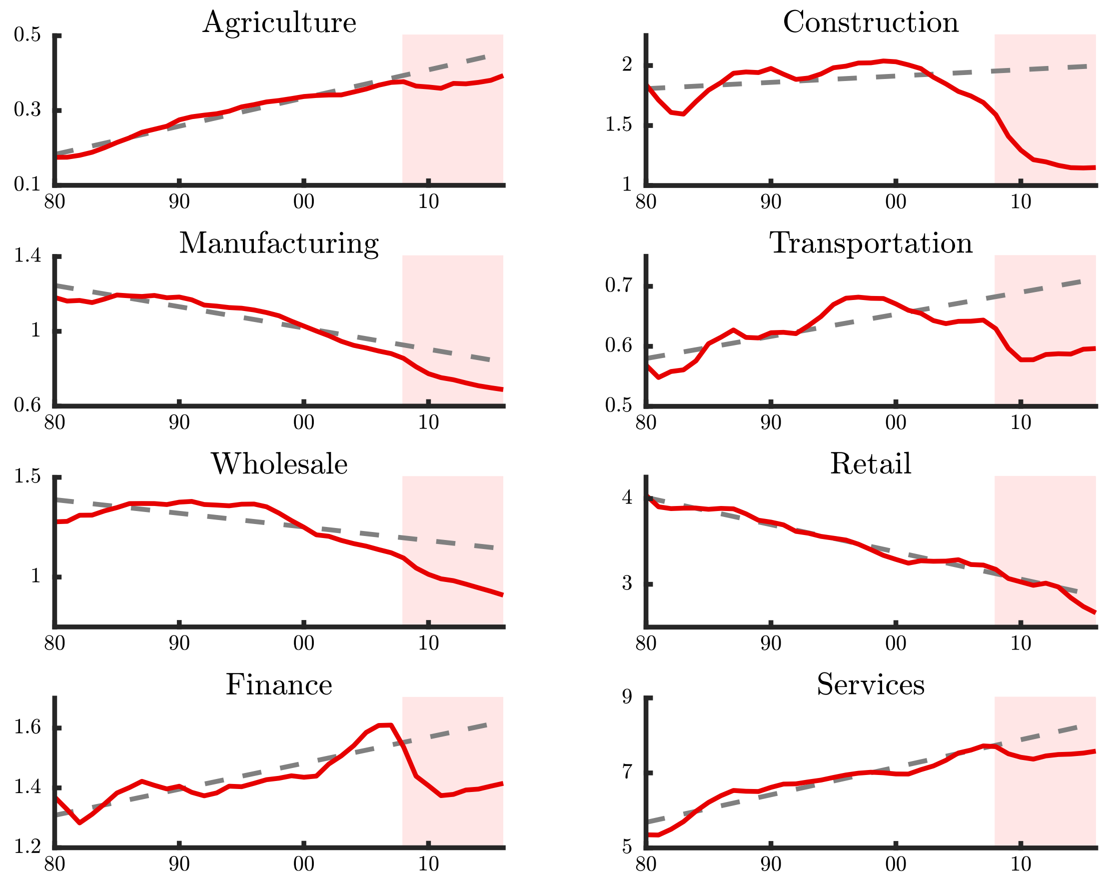


Figure 19: **Number of Firms per Sector (per capita): 1980-2016**
Each panel panel shows the number of firms with at least one employee in each sector (from the US BDS) per 1000 people. For each series, the dashed grey line shows a linear trend computed over the 1980-2007 period.

C.2 Entry and Exit Rates

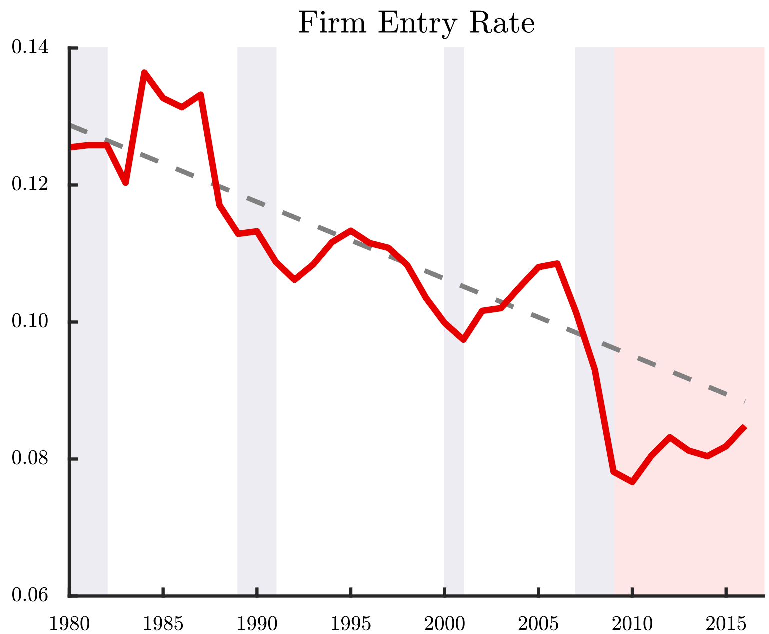


Figure 20: **US Firm Entry Rate: 1980-2017**
The entry rate is ratio of the number of startups to the number of active firms in the previous year (data is from the US Business Dynamic Statistics). The dashed grey line shows a linear trend computed for the 1980-2007 period.

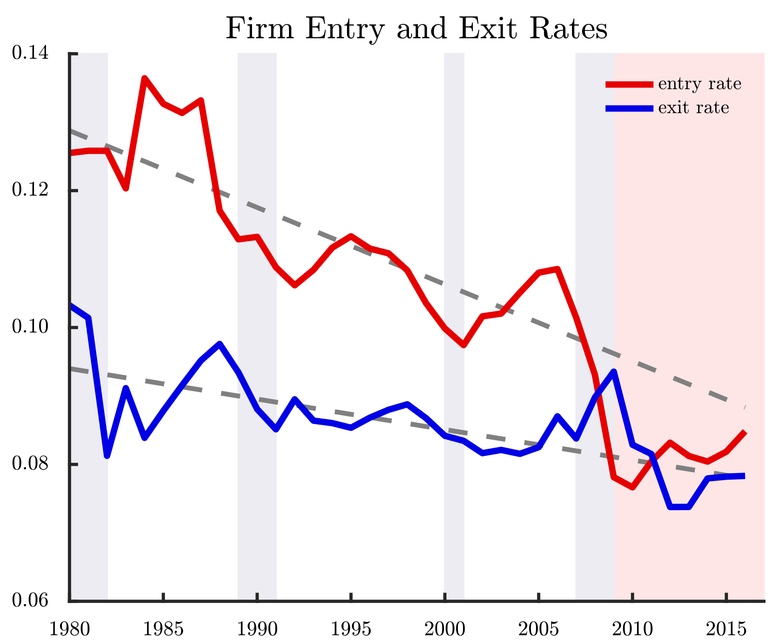
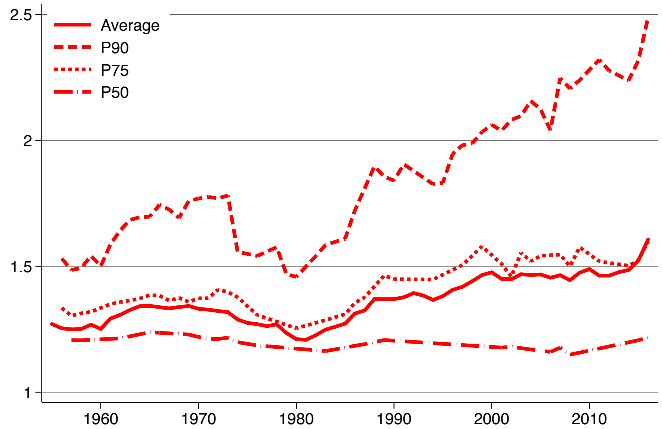


Figure 21: **US Firm Entry and Exit Rates: 1980-2017**
The entry (exit) rate is ratio of the number of startups (exiting firms) to the number of active firms in the previous year (data is from the US Business Dynamic Statistics). The dashed grey line shows a linear trend computed for the 1980-2007 period.

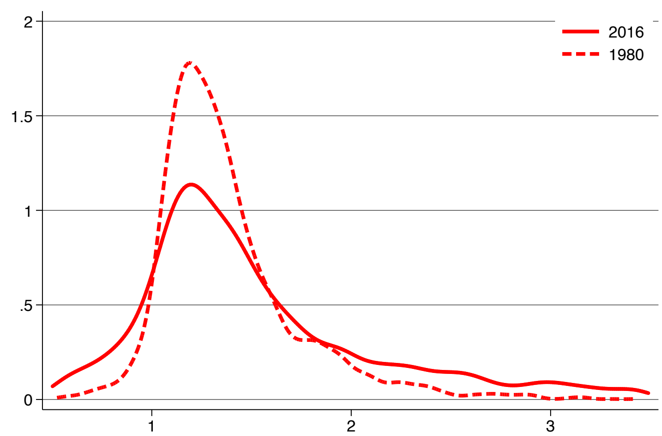
D Markups: De Loecker, Eeckhout and Unger (2018)



(a) Production Function PF1

Figure 8: Percentiles of the Markup Distribution

Figure 22: Percentiles of US Markup Distribution, from De Loecker, Eeckhout and Unger (2018)



(a) Production Function PF1

Figure 7: Distribution of Markups μ_{it}

Figure 23: US Markup Distribution, from De Loecker, Eeckhout and Unger (2018)

E Model Derivation and Proofs

E.1 Industry Equilibrium

E.1.1 Equilibrium Price

When n firms produce, we have a system of n first order conditions

$$p\left[1-(1-\rho)\,s^j\right]=\frac{\Theta}{\pi^j}$$

Dividing the first order condition of firm j by that of firm 1 we obtain

$$\begin{aligned}
&\frac{1-(1-\rho)\,s^j}{1-(1-\rho)\,s^1}=\frac{\pi^1}{\pi^j}\\
\Leftrightarrow\quad &1-(1-\rho)\,s^j=\frac{\pi^1}{\pi^j}\left[1-(1-\rho)\,s^1\right]\\
\Leftrightarrow\quad &s^j=\frac{1}{(1-\rho)}\left\{1-\frac{\pi^1}{\pi^j}\left[1-(1-\rho)\,s^1\right]\right\}
\end{aligned}$$

Note that

$$\begin{aligned}
&\sum_{k=1}^{n} s^k=1\\
\Leftrightarrow\quad &\sum_{k=1}^{n}\frac{1}{(1-\rho)}\left\{1-\frac{\pi^1}{\pi^k}\left[1-(1-\rho)\,s^1\right]\right\}=1\\
\Leftrightarrow\quad &n-\pi^1\left[1-(1-\rho)\,s^1\right]\sum_{k=1}^{n}\frac{1}{\pi^k}=1-\rho\\
\Leftrightarrow\quad &\frac{n-(1-\rho)}{\sum_{k=1}^{n}\frac{1}{\pi^k}}=\pi^1\left[1-(1-\rho)\,s^1\right]
\end{aligned}$$

Plugging the last equation into the first order condition of firm 1 we obtain

$$\begin{aligned}
&p\frac{n-(1-\rho)}{\sum_{k=1}^{n}\frac{1}{\pi^k}}=\Theta\\
\Leftrightarrow\quad &p=\frac{\sum_{k=1}^{n}\frac{1}{\pi^k}}{n-(1-\rho)}\Theta
\end{aligned}$$

Total output is hence equal to

$$\begin{aligned}
&y=p^{-\frac{1}{1-\rho}}Y\\
\Leftrightarrow\quad &y=\left[\frac{\sum_{k=1}^{n}\frac{1}{\pi^k}}{n-(1-\rho)}\Theta\right]^{-\frac{1}{1-\rho}}Y
\end{aligned}$$

E.1.2 Market Shares

Plugging the previous equation into the first order condition of firm j we have

$$\begin{aligned} 1-(1-\rho)\,s^j &= \frac{n-(1-\rho)}{\sum_{k=1}^{n}\frac{1}{\pi^k}}\frac{1}{\pi^j} \\ \Leftrightarrow\quad s^j &= \frac{1}{1-\rho}\left[1-\frac{n-(1-\rho)}{\sum_{k=1}^{n}\frac{1}{\pi^k}}\frac{1}{\pi^j}\right] \end{aligned}$$

It is easy to verify that each firm's market share decreases in the total number of active firms. To see this, suppose that the number of firms increases from n to $n+1$. The new entrant will have a market share

$$s^{n+1} = \frac{1}{1-\rho}\left[1-\frac{n+1-(1-\rho)}{\sum_{k=1}^{n+1}\frac{1}{\pi^k}}\frac{1}{\pi^{n+1}}\right]$$

which is non-negative provided that

$$\pi^{n+1}\sum_{k=1}^{n+1}\frac{1}{\pi^k} > n+1-(1-\rho) \tag{16}$$

and below one given that

$$\pi^{n+1}\sum_{k=1}^{n+1}\frac{1}{\pi^k} < \frac{1}{\rho}\left[n+1-(1-\rho)\right] \tag{17}$$

If we compare the market share of firm j when there n and $n+1$ firms in the market, we have

$$\begin{aligned} & s^j\,|n+1 < s^j\,|n \\ \Leftrightarrow\quad & \frac{1}{1-\rho}\left[1-\frac{n+1-(1-\rho)}{\sum_{k=1}^{n+1}\frac{1}{\pi^k}}\frac{1}{\pi^j}\right] < \frac{1}{1-\rho}\left[1-\frac{n-(1-\rho)}{\sum_{k=1}^{n}\frac{1}{\pi^k}}\frac{1}{\pi^j}\right] \\ \Leftrightarrow\quad & \frac{n-(1-\rho)}{\sum_{k=1}^{n}\frac{1}{\pi^k}} < \frac{n+1-(1-\rho)}{\sum_{k=1}^{n+1}\frac{1}{\pi^k}} \\ \Leftrightarrow\quad & [n-(1-\rho)]\left(\frac{1}{\pi^{n+1}}+\sum_{k=1}^{n}\frac{1}{\pi^k}\right) < [n+1-(1-\rho)]\sum_{k=1}^{n}\frac{1}{\pi^k} \\ \Leftrightarrow\quad & [n-(1-\rho)]\frac{1}{\pi^{n+1}} < \sum_{k=1}^{n}\frac{1}{\pi^k} \\ \Leftrightarrow\quad & \pi^{n+1}\sum_{k=1}^{n+1}\frac{1}{\pi^k} > n-(1-\rho) \end{aligned}$$

Note that the last condition is implied by (16).

E.1.3 Markups

Note that the markup of firm j is equal to,

$$\Pi^F\left(\underset{\downarrow}{\pi}, \underset{\downarrow}{n_{i,t}}, \underset{\downarrow}{\Theta_t}, \underset{\uparrow}{Y_t}\right)$$

$$\mu_{i,t}^{j} := \frac{p_{i,t}}{\left(\Theta_t/\pi_{i,t}^{j}\right)} = \frac{\sum_{k=1}^{n_{i,t}} \frac{\pi^j}{\pi^k}}{n_{i,t} - (1-\rho)}$$

From this expression we clearly see that

$$\frac{\partial \mu_{i,t}^{j}}{\partial \pi^k} = \begin{cases} > 0 & \text{if } k = j \\ < 0 & \text{if } k \neq j \end{cases}$$

We therefore have that the average markup is

$$\bar{\mu}_{i,t} = \frac{\frac{1}{n_{i,t}} \sum_{j=1}^{n_{i,t}} \sum_{k=1}^{n_{i,t}} \frac{\pi^j}{\pi^k}}{n_{i,t} - (1-\rho)}$$

And the variance is

$$\operatorname{var}\left(\mu_{i,t}\right) = \frac{1}{n_{i,t}} \sum_{j=1}^{n_{i,t}} \left(\frac{\sum_{k=1}^{n_{i,t}} \frac{\pi^j}{\pi^k}}{n_{i,t} - (1-\rho)} - \frac{\frac{1}{n_{i,t}} \sum_{h=1}^{n_{i,t}} \sum_{k=1}^{n_{i,t}} \frac{\pi^h}{\pi^k}}{n_{i,t} - (1-\rho)} \right)^2$$

E.1.4 Profits

When there are n active firms, type π^j makes production profits

$$\begin{aligned}
\Pi\left(\pi^j, n, \mathcal{F}, \Theta, Y\right) &= \left(p - \frac{\Theta}{\pi^j}\right) s^j y \\
&= \left(\frac{\sum_{k=1}^{n} \frac{1}{\pi^k}}{n - (1-\rho)} \Theta - \frac{\Theta}{\pi^j} \right) \underbrace{\frac{1}{1-\rho} \left[1 - \frac{n-(1-\rho)}{\sum_{k=1}^{n} \frac{1}{\pi^k}} \frac{1}{\pi^j} \right]}_{s^j} \underbrace{\left[\frac{\sum_{k=1}^{n} \frac{1}{\pi^k}}{n - (1-\rho)} \Theta \right]^{-\frac{1}{1-\rho}}}_{y} Y \\
&= \underbrace{\frac{1}{1-\rho} \left(\frac{\sum_{k=1}^{n} \frac{1}{\pi^k}}{n - (1-\rho)} - \frac{1}{\pi^j} \right) \left[1 - \frac{n-(1-\rho)}{\sum_{k=1}^{n} \frac{1}{\pi^k}} \frac{1}{\pi^j} \right] \left[\frac{\sum_{k=1}^{n} \frac{1}{\pi^k}}{n - (1-\rho)} \right]^{-\frac{1}{1-\rho}}}_{:=\Lambda(\pi^j, n)} \Theta^{-\frac{\rho}{1-\rho}} Y
\end{aligned}$$

E.2 Aggregate TFP

Let s^j denote the market share of firm j and let is^j denote its input share. Note that firm j produces $\frac{s^j}{s^1}$ as much output as firm 1 and uses $\frac{s^j}{s^1}\frac{\pi^1}{\pi^j}$ as many inputs. We have that

$$\begin{aligned} & \textstyle\sum_{k=1}^n is^k = 1 \\ \Leftrightarrow \quad & is^1 \frac{\pi^1}{s^1} \textstyle\sum_{k=1}^n \frac{s^k}{\pi^k} = 1 \\ \Leftrightarrow \quad & is^1 = \frac{s^1}{\pi^1} \left(\textstyle\sum_{k=1}^n \frac{s^k}{\pi^k} \right)^{-1} \end{aligned}$$

Note that we can write aggregate output as

$$\begin{aligned} & Y_t = \textstyle\sum_{k=1}^n y^k \\ \Leftrightarrow \quad & Y_t = \textstyle\sum_{k=1}^n \pi^k \left(is^k L \right)^{1-\alpha} \left(is^k K \right)^{\alpha} \\ \Leftrightarrow \quad & Y_t = \left(\textstyle\sum_{k=1}^n \pi^k is^k \right) L^{1-\alpha} K^{\alpha} \\ \Leftrightarrow \quad & Y_t = \left(\textstyle\sum_{k=1}^n \pi^k \frac{s^k}{s^1} \frac{\pi^1}{\pi^k} is^1 \right) L^{1-\alpha} K^{\alpha} \\ \Leftrightarrow \quad & Y_t = \left(\frac{\pi^1}{s^1} is^1 \underbrace{\sum_{k=1}^n s^k}_{=1} \right) L^{1-\alpha} K^{\alpha} \\ \Leftrightarrow \quad & Y_t = \left(\textstyle\sum_{k=1}^n \frac{s^k}{\pi^k} \right)^{-1} L^{1-\alpha} K^{\alpha} \\ \Leftrightarrow \quad & Y_t = \underbrace{\left\{ \sum_{k=1}^n \frac{1}{\pi^k} \frac{1}{1-\rho} \left[1 - \frac{n-(1-\rho)}{\sum_{h=1}^n \frac{1}{\pi^h}} \frac{1}{\pi^k} \right] \right\}^{-1}}_{:=\Phi(n_t)} L^{1-\alpha} K^{\alpha} \end{aligned}$$

We want to prove that

$$\sum_{k=1}^{n+1}\frac{1}{\pi^k}\frac{1}{1-\rho}\left[1-\frac{n+1-(1-\rho)}{\sum_{h=1}^{n+1}\frac{1}{\pi^h}}\frac{1}{\pi^k}\right]>\sum_{k=1}^{n}\frac{1}{\pi^k}\frac{1}{1-\rho}\left[1-\frac{n-(1-\rho)}{\sum_{h=1}^{n}\frac{1}{\pi^h}}\frac{1}{\pi^k}\right]$$

$$\Leftrightarrow\quad \sum_{k=1}^{n}\frac{1}{\pi^k}+\frac{1}{\pi^{n+1}}-\sum_{k=1}^{n+1}\frac{1}{\pi^k}\left[\frac{n+1-(1-\rho)}{\sum_{h=1}^{n+1}\frac{1}{\pi^h}}\frac{1}{\pi^k}\right]>\sum_{k=1}^{n}\frac{1}{\pi^k}-\sum_{k=1}^{n}\frac{1}{\pi^k}\left[\frac{n-(1-\rho)}{\sum_{h=1}^{n}\frac{1}{\pi^h}}\frac{1}{\pi^k}\right]$$

$$\Leftrightarrow\quad \frac{1}{\pi^{n+1}}\sum_{h=1}^{n+1}\frac{1}{\pi^h}-\sum_{k=1}^{n+1}\frac{1}{\pi^k}\left[\frac{n+1-(1-\rho)}{\pi^k}\right]>-\sum_{k=1}^{n}\frac{1}{\pi^k}\left[\frac{\sum_{h=1}^{n+1}\frac{1}{\pi^h}}{\sum_{h=1}^{n}\frac{1}{\pi^h}}\frac{n-(1-\rho)}{\pi^k}\right]$$

$$\Leftrightarrow\quad \frac{1}{\pi^{n+1}}\sum_{h=1}^{n+1}\frac{1}{\pi^h}-\sum_{k=1}^{n}\frac{1}{\pi^k}\left[\frac{n+1-(1-\rho)}{\pi^k}\right]-\frac{1}{\pi^{n+1}}\left[\frac{n+1-(1-\rho)}{\pi^{n+1}}\right]>-\sum_{k=1}^{n}\frac{1}{\pi^k}\left[\frac{\sum_{h=1}^{n}\frac{1}{\pi^h}+\frac{1}{\pi^{n+1}}}{\sum_{h=1}^{n}\frac{1}{\pi^h}}\frac{n-(1-\rho)}{\pi^k}\right]$$

$$\Leftrightarrow\quad \frac{1}{\pi^{n+1}}\sum_{h=1}^{n+1}\frac{1}{\pi^h}-\sum_{k=1}^{n}\left(\frac{1}{\pi^k}\right)^2-\frac{1}{\pi^{n+1}}\left[\frac{n+1-(1-\rho)}{\pi^{n+1}}\right]>-\sum_{k=1}^{n}\frac{1}{\pi^k}\left[\frac{\frac{1}{\pi^{n+1}}}{\sum_{h=1}^{n}\frac{1}{\pi^h}}\frac{n-(1-\rho)}{\pi^k}\right]$$

$$\Leftrightarrow\quad \frac{1}{\pi^{n+1}}\left[\sum_{h=1}^{n+1}\frac{1}{\pi^h}-\frac{n+1-(1-\rho)}{\pi^{n+1}}\right]-\sum_{k=1}^{n}\left(\frac{1}{\pi^k}\right)^2>-\sum_{k=1}^{n}\frac{1}{\pi^k}\left[\frac{\frac{1}{\pi^{n+1}}}{\sum_{h=1}^{n}\frac{1}{\pi^h}}\frac{n-(1-\rho)}{\pi^k}\right]$$

$$\Leftrightarrow\quad \frac{1}{\pi^{n+1}}\left[\sum_{h=1}^{n+1}\frac{1}{\pi^h}-\frac{n+1-(1-\rho)}{\pi^{n+1}}\right]>\sum_{k=1}^{n}\left(\frac{1}{\pi^k}\right)^2\left[1-\frac{\frac{1}{\pi^{n+1}}}{\sum_{h=1}^{n}\frac{1}{\pi^h}}[n-(1-\rho)]\right]$$

$$\Leftrightarrow\quad \left(\frac{1}{\pi^{n+1}}\right)^2\left\{\sum_{h=1}^{n+1}\frac{\pi^{n+1}}{\pi^h}-[n+1-(1-\rho)]\right\}>\sum_{k=1}^{n}\left(\frac{1}{\pi^k}\right)^2\left[1-\frac{1}{\sum_{h=1}^{n}\frac{\pi^{n+1}}{\pi^h}}[n-(1-\rho)]\right]$$

$$\Leftrightarrow\quad \left\{\sum_{h=1}^{n+1}\frac{\pi^{n+1}}{\pi^h}-[n+1-(1-\rho)]\right\}>\sum_{k=1}^{n}\left(\frac{\pi^{n+1}}{\pi^k}\right)^2\left[1-\frac{1}{\sum_{h=1}^{n}\frac{\pi^{n+1}}{\pi^h}}[n-(1-\rho)]\right]$$

$$\Leftrightarrow\quad \left\{\sum_{h=1}^{n+1}\frac{\pi^{n+1}}{\pi^h}-[n+1-(1-\rho)]\right\}>\frac{\sum_{k=1}^{n}\left(\frac{\pi^{n+1}}{\pi^k}\right)^2}{\sum_{h=1}^{n}\frac{\pi^{n+1}}{\pi^h}}\left\{\sum_{h=1}^{n}\frac{\pi^{n+1}}{\pi^h}-[n-(1-\rho)]\right\}$$

$$\Leftrightarrow\quad \left\{\sum_{h=1}^{n}\frac{\pi^{n+1}}{\pi^h}+1-[n+1-(1-\rho)]\right\}>\frac{\sum_{k=1}^{n}\left(\frac{\pi^{n+1}}{\pi^k}\right)^2}{\sum_{h=1}^{n}\frac{\pi^{n+1}}{\pi^h}}\left\{\sum_{h=1}^{n}\frac{\pi^{n+1}}{\pi^h}-[n-(1-\rho)]\right\}$$

$$\Leftrightarrow\quad \left\{\sum_{h=1}^{n}\frac{\pi^{n+1}}{\pi^h}-[n-(1-\rho)]\right\}>\frac{\sum_{k=1}^{n}\left(\frac{\pi^{n+1}}{\pi^k}\right)^2}{\sum_{h=1}^{n}\frac{\pi^{n+1}}{\pi^h}}\left\{\sum_{h=1}^{n}\frac{\pi^{n+1}}{\pi^h}-[n-(1-\rho)]\right\}$$

$$\Leftrightarrow\quad \frac{\sum_{k=1}^{n}\left(\frac{\pi^{n+1}}{\pi^k}\right)^2}{\sum_{k=1}^{n}\frac{\pi^{n+1}}{\pi^k}}<1$$

The last inequality is always satisfied provided that $\pi^{n+1}<\pi^k\ \forall k\leq n$

E.3 Aggregate Factor Share

E.3.1 Proof of Lemma 3

Recall that the aggregate factor share can be written as

$$\Omega(\mathcal{F}_\pi, n_t) = \frac{n-(1-\rho)}{\sum_{k=1}^{n}\frac{1}{\pi^k}}\frac{1}{\Phi(\mathcal{F}_\pi, n_t)}$$

As we have seen before, aggregate TFP $\Phi(\mathcal{F}_\pi, n_t)$ is decreasing in n_t. To show that $\Omega(\mathcal{F}_\pi, n_t)$ increases in n_t, it suffices to show that

$$\begin{aligned}
& \frac{n+1-(1-\rho)}{\sum_{k=1}^{n}\frac{1}{\pi^k}+\frac{1}{\pi^{n+1}}} > \frac{n-(1-\rho)}{\sum_{k=1}^{n}\frac{1}{\pi^k}} \\
\Leftrightarrow \quad & \frac{n+1-(1-\rho)}{n-(1-\rho)} > \frac{\sum_{k=1}^{n}\frac{1}{\pi^k}+\frac{1}{\pi^{n+1}}}{\sum_{k=1}^{n}\frac{1}{\pi^k}} \\
\Leftrightarrow \quad & \frac{1}{n-(1-\rho)} > \frac{\frac{1}{\pi^{n+1}}}{\sum_{k=1}^{n}\frac{1}{\pi^k}} \\
\Leftrightarrow \quad & \pi^{n+1}\sum_{k=1}^{n}\frac{1}{\pi^k} > n-(1-\rho)
\end{aligned}$$

The last condition is implied by (16).

E.3.2 Proof of Lemma 4

Note that the aggregate factor can be written as

$$\Omega(\mathcal{F}_\pi, n_t) = \left(\frac{n}{1-\rho}-1\right)\left\{\sum_{k=1}^{n}\frac{\frac{1}{\pi^k}}{\sum_{h=1}^{n}\frac{1}{\pi^h}}\left[1-[n-(1-\rho)]\frac{\frac{1}{\pi^k}}{\sum_{h=1}^{n}\frac{1}{\pi^h}}\right]\right\}$$

Take some $j < n$ such that $\pi^j \geq \frac{1}{n}\sum_{h=1}^{n}\pi^h$. First note that we must have that

$$\frac{\frac{1}{\pi^j}}{\frac{1}{n}\sum_{h=1}^{n}\frac{1}{\pi^h}} \leq 1$$

To see this, note that

$$
\begin{aligned}
& \frac{1}{\pi^j} \leq \frac{1}{n}\sum_{h=1}^n \frac{1}{\pi^h} \\
\Leftrightarrow \quad & 1 \leq \frac{1}{n}\sum_{h=1}^n \frac{\pi^j}{\pi^h} \\
\Leftrightarrow \quad & 1 \leq \frac{1}{n}\sum_{h=1}^n \frac{\pi^j}{\frac{1}{n}\sum_{k=1}^n \pi^k} \frac{\frac{1}{n}\sum_{k=1}^n \pi^k}{\pi^h} \\
\Leftrightarrow \quad & 1 \leq \underbrace{\frac{\pi^j}{\frac{1}{n}\sum_{k=1}^n \pi^k}}_{\geq 1} \underbrace{\frac{1}{n}\sum_{h=1}^n \frac{\frac{1}{n}\sum_{k=1}^n \pi^k}{\pi^h}}_{\geq 1}
\end{aligned}
$$

Now suppose that π^j increases to $\tilde{\pi}^j > \pi^j$. We want to show that

$$
\begin{aligned}
& \Omega\left(\tilde{\mathcal{F}}_\pi, n_t\right) < \Omega\left(\mathcal{F}_\pi, n_t\right) \\
\Leftrightarrow \quad & \sum_{k=1}^n \frac{\frac{1}{\tilde{\pi}^k}}{\sum_{h=1}^n \frac{1}{\tilde{\pi}^h}} \left[1 - [n-(1-\rho)] \frac{\frac{1}{\tilde{\pi}^k}}{\sum_{h=1}^n \frac{1}{\tilde{\pi}^h}}\right] < \sum_{k=1}^n \frac{\frac{1}{\pi^k}}{\sum_{h=1}^n \frac{1}{\pi^h}} \left[1 - [n-(1-\rho)] \frac{\frac{1}{\pi^k}}{\sum_{h=1}^n \frac{1}{\pi^h}}\right] \\
\Leftrightarrow \quad & \sum_{k=1}^n \frac{1}{\tilde{\pi}^k}\left[\sum_{h=1}^n \frac{1}{\tilde{\pi}^h} - [n-(1-\rho)]\frac{1}{\tilde{\pi}^k}\right] < \frac{\sum_{h=1}^n \frac{1}{\tilde{\pi}^h}}{\sum_{h=1}^n \frac{1}{\pi^h}} \sum_{k=1}^n \frac{1}{\pi^k}\left[\sum_{h=1}^n \frac{1}{\tilde{\pi}^h} - [n-(1-\rho)]\frac{\sum_{h=1}^n \frac{1}{\tilde{\pi}^h}}{\sum_{h=1}^n \frac{1}{\pi^h}}\frac{1}{\pi^k}\right] \\
\Leftrightarrow \quad & \sum_{k=1}^n \frac{1}{\tilde{\pi}^k}\left[\sum_{h=1}^n \frac{1}{\tilde{\pi}^h} - [n-(1-\rho)]\frac{1}{\tilde{\pi}^k}\right] < \left[1 + \frac{\frac{1}{\tilde{\pi}^j} - \frac{1}{\pi^j}}{\sum_{h=1}^n \frac{1}{\pi^h}}\right] \sum_{k=1}^n \frac{1}{\pi^k}\left\{\sum_{h=1}^n \frac{1}{\tilde{\pi}^h} - [n-(1-\rho)]\left[1 + \frac{\frac{1}{\tilde{\pi}^j} - \frac{1}{\pi^j}}{\sum_{h=1}^n \frac{1}{\pi^h}}\right]\frac{1}{\pi^k}\right\} \\
\Leftrightarrow \quad & \sum_{k=1}^n \frac{1}{\tilde{\pi}^k}\left[\sum_{h=1}^n \frac{1}{\tilde{\pi}^h} - [n-(1-\rho)]\frac{1}{\tilde{\pi}^k}\right] < \sum_{k=1}^n \frac{1}{\pi^k}\left[\sum_{h=1}^n \frac{1}{\tilde{\pi}^h} - [n-(1-\rho)]\frac{1}{\pi^k}\right] + \\
& + \frac{\frac{1}{\tilde{\pi}^j} - \frac{1}{\pi^j}}{\sum_{h=1}^n \frac{1}{\pi^h}} \sum_{k=1}^n \frac{1}{\pi^k}\left\{\sum_{h=1}^n \frac{1}{\tilde{\pi}^h} - [n-(1-\rho)]\left[1 + \frac{\frac{1}{\tilde{\pi}^j} - \frac{1}{\pi^j}}{\sum_{h=1}^n \frac{1}{\pi^h}}\right]\frac{1}{\pi^k}\right\} - \sum_{k=1}^n \frac{1}{\pi^k}\left\{[n-(1-\rho)]\frac{\frac{1}{\tilde{\pi}^j} - \frac{1}{\pi^j}}{\sum_{h=1}^n \frac{1}{\pi^h}}\frac{1}{\pi^k}\right\} \\
\Leftrightarrow \quad & \frac{1}{\tilde{\pi}^j}\left[\sum_{h=1}^n \frac{1}{\tilde{\pi}^h} - [n-(1-\rho)]\frac{1}{\tilde{\pi}^j}\right] < \frac{1}{\pi^j}\left[\sum_{h=1}^n \frac{1}{\tilde{\pi}^h} - [n-(1-\rho)]\frac{1}{\pi^j}\right] + \\
& + \frac{\frac{1}{\tilde{\pi}^j} - \frac{1}{\pi^j}}{\sum_{h=1}^n \frac{1}{\pi^h}} \sum_{k=1}^n \frac{1}{\pi^k}\left\{\sum_{h=1}^n \frac{1}{\tilde{\pi}^h} - [n-(1-\rho)]\left[1 + \frac{\frac{1}{\tilde{\pi}^j} - \frac{1}{\pi^j}}{\sum_{h=1}^n \frac{1}{\pi^h}}\right]\frac{1}{\pi^k}\right\} - [n-(1-\rho)]\frac{\frac{1}{\tilde{\pi}^j} - \frac{1}{\pi^j}}{\sum_{h=1}^n \frac{1}{\pi^h}}\sum_{k=1}^n\left(\frac{1}{\pi^k}\right)^2 \\
\Leftrightarrow \quad & \left(\frac{1}{\tilde{\pi}^j} - \frac{1}{\pi^j}\right)\sum_{h=1}^n \frac{1}{\tilde{\pi}^h} - [n-(1-\rho)]\left[\left(\frac{1}{\tilde{\pi}^j}\right)^2 - \left(\frac{1}{\pi^j}\right)^2\right] < \\
& < \frac{\frac{1}{\tilde{\pi}^j} - \frac{1}{\pi^j}}{\sum_{h=1}^n \frac{1}{\pi^h}} \sum_{k=1}^n \frac{1}{\pi^k}\left\{\sum_{h=1}^n \frac{1}{\tilde{\pi}^h} - [n-(1-\rho)]\left[1 + \frac{\frac{1}{\tilde{\pi}^j} - \frac{1}{\pi^j}}{\sum_{h=1}^n \frac{1}{\pi^h}}\right]\frac{1}{\pi^k}\right\} - [n-(1-\rho)]\frac{\frac{1}{\tilde{\pi}^j} - \frac{1}{\pi^j}}{\sum_{h=1}^n \frac{1}{\pi^h}}\sum_{k=1}^n\left(\frac{1}{\pi^k}\right)^2 \\
\Leftrightarrow \quad & \sum_{h=1}^n \frac{1}{\tilde{\pi}^h} - [n-(1-\rho)]\left(\frac{1}{\pi^j} + \frac{1}{\tilde{\pi}^j}\right) > \\
& > \frac{1}{\sum_{h=1}^n \frac{1}{\pi^h}} \sum_{k=1}^n \frac{1}{\pi^k}\left\{\sum_{h=1}^n \frac{1}{\tilde{\pi}^h} - [n-(1-\rho)]\left[1 + \frac{\frac{1}{\tilde{\pi}^j} - \frac{1}{\pi^j}}{\sum_{h=1}^n \frac{1}{\pi^h}}\right]\frac{1}{\pi^k}\right\} - [n-(1-\rho)]\frac{\sum_{k=1}^n\left(\frac{1}{\pi^k}\right)^2}{\sum_{h=1}^n \frac{1}{\pi^h}}
\end{aligned}
$$

$$\Leftrightarrow\quad \textstyle\sum_{h=1}^{n}\frac{1}{\tilde{\pi}^h}-[n-(1-\rho)]\left(\frac{1}{\pi^j}+\frac{1}{\tilde{\pi}^j}\right)>\underbrace{\frac{\sum_{k=1}^{n}\frac{1}{\pi^k}}{\sum_{h=1}^{n}\frac{1}{\pi^h}}}_{=1}\sum_{h=1}^{n}\frac{1}{\tilde{\pi}^h}-\frac{n-(1-\rho)}{\sum_{h=1}^{n}\frac{1}{\pi^h}}\left[2+\frac{\frac{1}{\tilde{\pi}^j}-\frac{1}{\pi^j}}{\sum_{h=1}^{n}\frac{1}{\pi^h}}\right]\sum_{k=1}^{n}\left(\frac{1}{\pi^k}\right)^2$$

$$\Leftrightarrow\quad -[n-(1-\rho)]\left(\frac{1}{\pi^j}+\frac{1}{\tilde{\pi}^j}\right)>-\frac{n-(1-\rho)}{\sum_{h=1}^{n}\frac{1}{\pi^h}}\left[2+\frac{\frac{1}{\tilde{\pi}^j}-\frac{1}{\pi^j}}{\sum_{h=1}^{n}\frac{1}{\pi^h}}\right]\sum_{k=1}^{n}\left(\frac{1}{\pi^k}\right)^2$$

$$\Leftrightarrow\quad \frac{1}{\pi^j}+\frac{1}{\tilde{\pi}^j}<\frac{1}{\sum_{h=1}^{n}\frac{1}{\pi^h}}\left[2+\frac{\frac{1}{\tilde{\pi}^j}-\frac{1}{\pi^j}}{\sum_{h=1}^{n}\frac{1}{\pi^h}}\right]\sum_{k=1}^{n}\left(\frac{1}{\pi^k}\right)^2$$

$$\Leftrightarrow\quad 1+\frac{\pi^j}{\tilde{\pi}^j}<\frac{\sum_{k=1}^{n}\left(\frac{\pi^j}{\pi^k}\right)^2}{\sum_{h=1}^{n}\frac{\pi^j}{\pi^h}}\left(2+\frac{\frac{\pi^j}{\tilde{\pi}^j}-1}{\sum_{h=1}^{n}\frac{\pi^j}{\pi^h}}\right)$$

$$\Leftrightarrow\quad \tilde{\pi}^j+\pi^j<2\tilde{\pi}^j\frac{\sum_{k=1}^{n}\left(\frac{\pi^j}{\pi^k}\right)^2}{\sum_{h=1}^{n}\frac{\pi^j}{\pi^h}}-\left(\tilde{\pi}^j-\pi^j\right)\frac{\sum_{k=1}^{n}\left(\frac{\pi^j}{\pi^k}\right)^2}{\left(\sum_{h=1}^{n}\frac{\pi^j}{\pi^h}\right)^2}$$

$$\Leftrightarrow\quad \pi^j\left[1-\frac{\sum_{k=1}^{n}\left(\frac{\pi^j}{\pi^k}\right)^2}{\left(\sum_{h=1}^{n}\frac{\pi^j}{\pi^h}\right)^2}\right]<\tilde{\pi}^j\left[2\frac{\sum_{k=1}^{n}\left(\frac{\pi^j}{\pi^k}\right)^2}{\sum_{h=1}^{n}\frac{\pi^j}{\pi^h}}-\frac{\sum_{k=1}^{n}\left(\frac{\pi^j}{\pi^k}\right)^2}{\left(\sum_{h=1}^{n}\frac{\pi^j}{\pi^h}\right)^2}-1\right]$$

$$\Leftrightarrow\quad \frac{\pi^j}{\tilde{\pi}^j}<\frac{2\frac{\sum_{k=1}^{n}\left(\frac{\pi^j}{\pi^k}\right)^2}{\sum_{h=1}^{n}\frac{\pi^j}{\pi^h}}-\frac{\sum_{k=1}^{n}\left(\frac{\pi^j}{\pi^k}\right)^2}{\left(\sum_{h=1}^{n}\frac{\pi^j}{\pi^h}\right)^2}-1}{1-\frac{\sum_{k=1}^{n}\left(\frac{\pi^j}{\pi^k}\right)^2}{\left(\sum_{h=1}^{n}\frac{\pi^j}{\pi^h}\right)^2}}$$

It only suffices to show that the right hand side of the above inequality is greater than one

$$2\sum_{k=1}^{n}\left(\frac{\pi^j}{\pi^k}\right)^2\left(\sum_{h=1}^{n}\frac{\pi^j}{\pi^h}\right)-\sum_{k=1}^{n}\left(\frac{\pi^j}{\pi^k}\right)^2-\left(\sum_{h=1}^{n}\frac{\pi^j}{\pi^h}\right)^2>\left(\sum_{h=1}^{n}\frac{\pi^j}{\pi^h}\right)^2-\sum_{k=1}^{n}\left(\frac{\pi^j}{\pi^k}\right)^2$$

$$\Leftrightarrow\quad 2\sum_{k=1}^{n}\left(\frac{\pi^j}{\pi^k}\right)^2\left(\sum_{h=1}^{n}\frac{\pi^j}{\pi^h}\right)-\left(\sum_{h=1}^{n}\frac{\pi^j}{\pi^h}\right)^2>\left(\sum_{h=1}^{n}\frac{\pi^j}{\pi^h}\right)^2$$

$$\Leftrightarrow\quad \sum_{k=1}^{n}\left(\frac{\pi^j}{\pi^k}\right)^2\left(\sum_{h=1}^{n}\frac{\pi^j}{\pi^h}\right)>\left(\sum_{h=1}^{n}\frac{\pi^j}{\pi^h}\right)^2$$

$$\Leftrightarrow\quad \sum_{k=1}^{n}\left(\frac{\pi^j}{\pi^k}\right)^2\left(\sum_{h=1}^{n}\frac{\pi^j}{\pi^h}\right)>\left(\sum_{h=1}^{n}\frac{\pi^j}{\pi^h}\right)\left(\sum_{h=1}^{n}\frac{\pi^j}{\pi^h}\right)$$

$$\Leftrightarrow\quad \sum_{h=1}^{n}\left(\frac{\pi^j}{\pi^h}\right)^2>\sum_{h=1}^{n}\frac{\pi^j}{\pi^h}$$

It is easy to prove that the last inequality is satisfied provided that $\pi^j \geq \frac{1}{n}\sum_{k=1}^n \pi^k$. Note that

$$
\begin{aligned}
& \sum_{h=1}^n \frac{\frac{1}{n}\sum_{k=1}^n \pi^k}{\pi^h}\frac{\pi^j}{\pi^h} > \sum_{h=1}^n \frac{\pi^j}{\pi^h} \\
\Leftrightarrow \quad & \left(\sum_{h=1}^n \frac{\pi^j}{\pi^h}\right) + \left(\sum_{h=1}^n \frac{\frac{1}{n}\sum_{k\neq h}^n \pi^k}{\pi^h}\frac{\pi^j}{\pi^h}\right) > \sum_{h=1}^n \frac{\pi^j}{\pi^h}
\end{aligned}
$$

E.4 Law of Motion

$$
\begin{aligned}
& Y_t = \Phi\left(\mathcal{F}_\pi, n_t\right)\left[\frac{(1-\alpha)\,\Theta\left(\mathcal{F}_\pi, n_t\right)}{\eta}\right]^{\frac{1-\alpha}{\nu+\alpha}} K_t^{\alpha\frac{1+\nu}{\nu+\alpha}} \\
\Leftrightarrow\quad & K_t^{\alpha\frac{1+\nu}{\nu+\alpha}} = \left[\Phi\left(\mathcal{F}_\pi, n_t\right)\right]^{-1}\left[\frac{\eta}{(1-\alpha)\,\Theta\left(\mathcal{F}_\pi, n_t\right)}\right]^{\frac{1-\alpha}{\nu+\alpha}} Y_t \\
\Leftrightarrow\quad & K_t = \left[\Phi\left(\mathcal{F}_\pi, n_t\right)\right]^{-\frac{\nu+\alpha}{\alpha(1+\nu)}}\left[\frac{\eta}{(1-\alpha)\,\Theta\left(\mathcal{F}_\pi, n_t\right)}\right]^{\frac{1-\alpha}{\alpha(1+\nu)}} Y_t^{\frac{\nu+\alpha}{\alpha(1+\nu)}}
\end{aligned}
$$

We also have that

$$
\begin{aligned}
& K_{t+1} = s_t Y_t \\
\Leftrightarrow\quad & K_{t+1} = \left(\beta_m R_{t+1}\right)^\gamma Y_t \\
\Leftrightarrow\quad & K_{t+1} = \left\{\beta_m\left(\frac{1-\alpha}{\eta}\right)^{\frac{1-\alpha}{\nu+\alpha}} \alpha\,\left[\Theta\left(\mathcal{F}_\pi, n_{t+1}\right)\right]^{\frac{\nu+1}{\nu+\alpha}} K_{t+1}^{-\nu\frac{1-\alpha}{\nu+\alpha}}\right\}^\gamma Y_t \\
\Leftrightarrow\quad & K_{t+1}^{1+\gamma\nu\frac{1-\alpha}{\nu+\alpha}} = \left\{\beta_m\left(\frac{1-\alpha}{\eta}\right)^{\frac{1-\alpha}{\nu+\alpha}} \alpha\,\left[\Theta\left(\mathcal{F}_\pi, n_{t+1}\right)\right]^{\frac{\nu+1}{\nu+\alpha}}\right\}^\gamma Y_t \\
\Leftrightarrow\quad & K_{t+1} = \left\{\beta_m\left(\frac{1-\alpha}{\eta}\right)^{\frac{1-\alpha}{\nu+\alpha}} \alpha\,\left[\Theta\left(\mathcal{F}_\pi, n_{t+1}\right)\right]^{\frac{\nu+1}{\nu+\alpha}}\right\}^{\frac{\gamma(\nu+\alpha)}{\nu+\alpha+\gamma\nu(1-\alpha)}} Y_t^{\frac{\nu+\alpha}{\nu+\alpha+\gamma\nu(1-\alpha)}} \\
\Leftrightarrow\quad & K_{t+1} = \iota_K \Gamma_K\left(\mathcal{F}_\pi, n_{t+1}\right) Y_t^{\varepsilon_K}
\end{aligned}
$$

And that

$$
\begin{aligned}
& K_{t+1}^{1+\gamma\nu\frac{1-\alpha}{\nu+\alpha}} = \left\{\beta_m\left(\frac{1-\alpha}{\eta}\right)^{\frac{1-\alpha}{\nu+\alpha}} \alpha\,\left[\Theta\left(\mathcal{F}_\pi, n_{t+1}\right)\right]^{\frac{\nu+1}{\nu+\alpha}}\right\}^\gamma Y_t \\
\Leftrightarrow\quad & \left\{\left[\Phi\left(\mathcal{F}_\pi, n_{t+1}\right)\right]^{-\frac{\nu+\alpha}{\alpha(1+\nu)}}\left[\frac{\eta}{(1-\alpha)\,\Theta\left(\mathcal{F}_\pi, n_{t+1}\right)}\right]^{\frac{1-\alpha}{\alpha(1+\nu)}} Y_{t+1}^{\frac{\nu+\alpha}{\alpha(1+\nu)}}\right\}^{\frac{\nu+\alpha+\gamma\nu(1-\alpha)}{\nu+\alpha}} = \left\{\beta_m\left(\frac{1-\alpha}{\eta}\right)^{\frac{1-\alpha}{\nu+\alpha}} \alpha\,\left[\Theta\left(\mathcal{F}_\pi, n_{t+1}\right)\right]^{\frac{\nu+1}{\nu+\alpha}}\right\}^\gamma Y_t \\
\Leftrightarrow\quad & \left[\Phi\left(\mathcal{F}_\pi, n_{t+1}\right)\right]^{-\frac{\nu+\alpha}{\alpha(1+\nu)}}\left[\frac{\eta}{(1-\alpha)\,\Theta\left(\mathcal{F}_\pi, n_{t+1}\right)}\right]^{\frac{1-\alpha}{\alpha(1+\nu)}} Y_{t+1}^{\frac{\nu+\alpha}{\alpha(1+\nu)}} = \left(\left\{\beta_m\left(\frac{1-\alpha}{\eta}\right)^{\frac{1-\alpha}{\nu+\alpha}} \alpha\,\left[\Theta\left(\mathcal{F}_\pi, n_{t+1}\right)\right]^{\frac{\nu+1}{\nu+\alpha}}\right\}^\gamma Y_t\right)^{\frac{\nu+\alpha}{\nu+\alpha+\gamma\nu(1-\alpha)}} \\
\Leftrightarrow\quad & \left[\Phi\left(\mathcal{F}_\pi, n_{t+1}\right)\right]^{-1}\left[\frac{\eta}{(1-\alpha)\,\Theta\left(\mathcal{F}_\pi, n_{t+1}\right)}\right]^{\frac{1-\alpha}{\nu+\alpha}} Y_{t+1} = \left\{\beta_m\left(\frac{1-\alpha}{\eta}\right)^{\frac{1-\alpha}{\nu+\alpha}} \alpha\,\left[\Theta\left(\mathcal{F}_\pi, n_{t+1}\right)\right]^{\frac{\nu+1}{\nu+\alpha}}\right\}^{\frac{\gamma\alpha(1+\nu)}{\nu+\alpha+\gamma\nu(1-\alpha)}} Y_t^{\frac{\alpha(1+\nu)}{\nu+\alpha+\gamma\nu(1-\alpha)}} \\
\Leftrightarrow\quad & Y_{t+1} = \Phi\left(\mathcal{F}_\pi, n_{t+1}\right)\left[\frac{(1-\alpha)\,\Theta\left(\mathcal{F}_\pi, n_{t+1}\right)}{\eta}\right]^{\frac{1-\alpha}{\nu+\alpha}}\left\{\beta_m\left(\frac{1-\alpha}{\eta}\right)^{\frac{1-\alpha}{\nu+\alpha}} \alpha\,\left[\Theta\left(\mathcal{F}_\pi, n_{t+1}\right)\right]^{\frac{\nu+1}{\nu+\alpha}}\right\}^{\frac{\gamma\alpha(1+\nu)}{\nu+\alpha+\gamma\nu(1-\alpha)}} Y_t^{\frac{\alpha(1+\nu)}{\nu+\alpha+\gamma\nu(1-\alpha)}} \\
\Leftrightarrow\quad & Y_{t+1} = \left(\frac{1-\alpha}{\eta}\right)^{\frac{1-\alpha}{\nu+\alpha}}\left[\beta_m\left(\frac{1-\alpha}{\eta}\right)^{\frac{1-\alpha}{\nu+\alpha}} \alpha\right]^{\frac{\gamma\alpha(1+\nu)}{\nu+\alpha+\gamma\nu(1-\alpha)}} \Phi\left(\mathcal{F}_\pi, n_{t+1}\right)\left[\Theta\left(\mathcal{F}_\pi, n_{t+1}\right)\right]^{\frac{1-\alpha}{\nu+\alpha}+\frac{\nu+1}{\nu+\alpha}\frac{\gamma\alpha(1+\nu)}{\nu+\alpha+\gamma\nu(1-\alpha)}} Y_t^{\frac{\alpha(1+\nu)}{\nu+\alpha+\gamma\nu(1-\alpha)}} \\
\Leftrightarrow\quad & Y_{t+1} = \left(\frac{1-\alpha}{\eta}\right)^{\frac{1-\alpha}{\nu+\alpha}\left[1+\frac{\gamma\alpha(1+\nu)}{\nu+\alpha+\gamma\nu(1-\alpha)}\right]}\left(\beta_m\alpha\right)^{\frac{\gamma\alpha(1+\nu)}{\nu+\alpha+\gamma\nu(1-\alpha)}} \Phi\left(\mathcal{F}_\pi, n_{t+1}\right)\left[\Theta\left(\mathcal{F}_\pi, n_{t+1}\right)\right]^{\frac{1}{\nu+\alpha}\left[1-\alpha+\frac{(\nu+1)\gamma\alpha(1+\nu)}{\nu+\alpha+\gamma\nu(1-\alpha)}\right]} Y_t^{\frac{\alpha(1+\nu)}{\nu+\alpha+\gamma\nu(1-\alpha)}} \\
\Leftrightarrow\quad & Y_{t+1} = \underbrace{\left(\frac{1-\alpha}{\eta}\right)^{\frac{2(1-\alpha)}{\nu[1+\gamma(1-\alpha)]+\alpha}}\left(\beta_m\alpha\right)^{\frac{\gamma\alpha(1+\nu)}{\nu[1+\gamma(1-\alpha)]+\alpha}}}_{:=\iota_Y} \underbrace{\Phi\left(\mathcal{F}_\pi, n_{t+1}\right)\left[\Theta\left(\mathcal{F}_\pi, n_{t+1}\right)\right]^{\frac{(1-\alpha)+\gamma(1+\nu\alpha)}{\nu[1+\gamma(1-\alpha)]+\alpha}}}_{:=\Gamma_Y\left(\mathcal{F}_\pi, n_{t+1}\right)} \underbrace{Y_t^{\frac{\alpha(1+\nu)}{\nu[1+\gamma(1-\alpha)]+\alpha}}}_{:=Y_t^{\varepsilon_Y}} \\
\Leftrightarrow\quad & Y_{t+1} = \iota_Y \Gamma_Y\left(\mathcal{F}_\pi, n_{t+1}\right) Y_t^{\varepsilon_Y}
\end{aligned}
$$

$$
\begin{aligned}
\exp_1 &= \tfrac{1-\alpha}{\nu+\alpha}\tfrac{\nu+\alpha+\gamma\nu(1-\alpha)+\gamma\alpha(1+\nu)}{\nu+\alpha+\gamma\nu(1-\alpha)} \\
&= \tfrac{1-\alpha}{\nu+\alpha}\tfrac{\nu+\alpha+\gamma[\nu(1-\alpha)+\alpha(1+\nu)]}{\nu+\alpha+\gamma\nu(1-\alpha)} \\
&= \tfrac{1-\alpha}{\nu+\alpha}\tfrac{2(\nu+\alpha)}{\nu+\alpha+\gamma\nu(1-\alpha)} \\
&= \tfrac{2(1-\alpha)}{\nu[1+\gamma(1-\alpha)]+\alpha} \\
\exp_2 &= \tfrac{1}{\nu+\alpha}\left[1-\alpha+\tfrac{(\nu+1)\gamma\alpha(1+\nu)}{\nu+\alpha+\gamma\nu(1-\alpha)}\right] \\
&= \tfrac{1}{\nu+\alpha}\tfrac{(\nu+\alpha)(1-\alpha)+\gamma\nu(1-\alpha)^2+\gamma\alpha(1+\nu)^2}{\nu+\alpha+\gamma\nu(1-\alpha)} \\
&= \tfrac{1}{\nu+\alpha}\tfrac{(\nu+\alpha)(1-\alpha)+\gamma\nu\left(1+\alpha^2-2\alpha\right)+\gamma\alpha\left(1+\nu^2+2\nu\right)}{\nu+\alpha+\gamma\nu(1-\alpha)} \\
&= \tfrac{1}{\nu+\alpha}\tfrac{(\nu+\alpha)(1-\alpha)+\gamma\nu\left(1+\alpha^2\right)+\gamma\alpha\left(1+\nu^2\right)}{\nu+\alpha+\gamma\nu(1-\alpha)} \\
&= \tfrac{1}{\nu+\alpha}\tfrac{(\nu+\alpha)(1-\alpha)+\gamma\nu+\gamma\nu\alpha^2+\gamma\alpha+\gamma\alpha\nu^2}{\nu+\alpha+\gamma\nu(1-\alpha)} \\
&= \tfrac{1}{\nu+\alpha}\tfrac{(\nu+\alpha)(1-\alpha)+(\nu+\alpha)\gamma(1+\nu\alpha)}{\nu+\alpha+\gamma\nu(1-\alpha)} \\
&= \tfrac{(1-\alpha)+\gamma(1+\nu\alpha)}{\nu+\alpha+\gamma\nu(1-\alpha)}
\end{aligned}
$$

$$
\begin{aligned}
\varepsilon_Y &= \tfrac{\alpha(1+\nu)}{\nu+\alpha+\gamma\nu(1-\alpha)} \\
\zeta &= \tfrac{(1-\alpha)+\gamma(1+\nu\alpha)}{\nu+\alpha+\gamma\nu(1-\alpha)}
\end{aligned}
$$

It is easy to see that $\Gamma_Y\left(\mathcal{F}_\pi, n_{t+1}\right)$ is increasing in n_{t+1}. Suppose that the economy moves from a symmetric equilibrium with n firms per industry to a symmetric equilibrium with $n+1$ firms per industry. Let π^j be a productivity type that produces in the second equilibrium, but not in the first one. The profits that π^j makes must be greater in the second equilibrium (otherwise, it would already be producing in the first equilibrium). This means that

$$
\Pi\left(\pi^j, n+1, \Theta^{'}, Y^{'}\right) > \Pi\left(\pi^j, n, \Theta, Y\right)
$$

As we have seen above, $\Pi(\cdot)$ decreases in n and Θ. Moreover, Θ increases in n, i.e. $\Theta^{'} > \Theta$. It therefore follows that output must necessarily be higher in the second equilibrium, i.e. $Y^{'} > Y$.

E.5 Conditions for Symmetric Equilibrium

As we have seen in Appendix E.1.4, when n firms produce in a given industry, and given aggregate output Y and factor costs Θ, type π^j makes production profits

$$\Pi\left(\pi^j, n, \Theta, Y\right) = \Lambda\left(\pi^j, n\right)\Theta^{-\frac{\rho}{1-\rho}}Y$$

where

$$\begin{aligned}
\Lambda\left(\pi^j, n\right) &\equiv \frac{1}{1-\rho}\left[\frac{\sum_{k=1}^{n}\frac{1}{\pi^k}}{n-(1-\rho)} - \frac{1}{\pi^j}\right]\left[1 - \frac{n-(1-\rho)}{\sum_{k=1}^{n}\frac{1}{\pi^k}}\frac{1}{\pi^j}\right]\left[\frac{n-(1-\rho)}{\sum_{k=1}^{n}\frac{1}{\pi^k}}\right]^{\frac{1}{1-\rho}} \\
&= \frac{1}{1-\rho}\left[\frac{\sum_{k=1}^{n}\frac{1}{\pi^k}}{n-(1-\rho)} - \frac{1}{\pi^j} - \frac{1}{\pi^j} + \frac{n-(1-\rho)}{\sum_{k=1}^{n}\frac{1}{\pi^k}}\left(\frac{1}{\pi^j}\right)^2\right]\left[\frac{n-(1-\rho)}{\sum_{k=1}^{n}\frac{1}{\pi^k}}\right]^{\frac{1}{1-\rho}} \\
&= \frac{1}{1-\rho}\left[\frac{\sum_{k=1}^{n}\frac{1}{\pi^k}}{n-(1-\rho)} - 2\frac{1}{\pi^j} + \frac{n-(1-\rho)}{\sum_{k=1}^{n}\frac{1}{\pi^k}}\left(\frac{1}{\pi^j}\right)^2\right]\left[\frac{n-(1-\rho)}{\sum_{k=1}^{n}\frac{1}{\pi^k}}\right]^{\frac{1}{1-\rho}} \\
&= \frac{1}{1-\rho}\frac{\sum_{k=1}^{n}\frac{1}{\pi^k}}{n-(1-\rho)}\left\{1 + -2\frac{1}{\pi^j}\frac{n-(1-\rho)}{\sum_{k=1}^{n}\frac{1}{\pi^k}} + \left[\frac{n-(1-\rho)}{\sum_{k=1}^{n}\frac{1}{\pi^k}}\frac{1}{\pi^j}\right]^2\right\}\left[\frac{n-(1-\rho)}{\sum_{k=1}^{n}\frac{1}{\pi^k}}\right]^{\frac{1}{1-\rho}} \\
&= \frac{1}{1-\rho}\left[1 - \frac{n-(1-\rho)}{\sum_{k=1}^{n}\frac{1}{\pi^k}}\frac{1}{\pi^j}\right]^2\left[\frac{n-(1-\rho)}{\sum_{k=1}^{n}\frac{1}{\pi^k}}\right]^{\frac{\rho}{1-\rho}}
\end{aligned}$$

It is immediate to see that $\Lambda\left(\pi^j, n\right)$ increases in π^j (i.e. given a fixed number of firms n, more productive firms make higher profits). We can also prove that $\Lambda\left(\pi^j, n\right)$ decreases in n (i.e. given a fixed productivity level π^j, having one additional competitor in the market reduces the current level of profits). Recall that $\frac{n-(1-\rho)}{\sum_{k=1}^{n}\frac{1}{\pi^k}}$ is increasing in n. We therefore have to prove that $\mathit{\Lambda}(\cdot)$ is decreasing in $\dfrac{n-(1-\rho)}{\sum_{k=1}^{n}\frac{1}{\pi^k}}$. This requires that

$$
\begin{aligned}
& 2\left[1-\frac{n-(1-\rho)}{\sum_{k=1}^{n}\frac{1}{\pi^k}}\frac{1}{\pi^j}\right]^{-1}\left(-\frac{1}{\pi^j}\right)+\tfrac{\rho}{1-\rho}\left[\frac{n-(1-\rho)}{\sum_{k=1}^{n}\frac{1}{\pi^k}}\right]^{-1}<0 \\
\Leftrightarrow\quad & \frac{\rho}{1-\rho}\left[1-\frac{n-(1-\rho)}{\sum_{k=1}^{n}\frac{1}{\pi^k}}\frac{1}{\pi^j}\right]<2\left[\frac{n-(1-\rho)}{\sum_{k=1}^{n}\frac{1}{\pi^k}}\right]\frac{1}{\pi^j} \\
\Leftrightarrow\quad & \frac{\rho}{1-\rho}<\left(2+\tfrac{\rho}{1-\rho}\right)\left[\frac{n-(1-\rho)}{\sum_{k=1}^{n}\frac{1}{\pi^k}}\right]\frac{1}{\pi^j} \\
\Leftrightarrow\quad & \rho<(2-\rho)\left[\frac{n-(1-\rho)}{\sum_{k=1}^{n}\frac{1}{\pi^k}}\right]\frac{1}{\pi^j} \\
\Leftrightarrow\quad & \pi^j\sum_{k=1}^{n}\frac{1}{\pi^k}<\frac{2-\rho}{\rho}\left[n-(1-\rho)\right]
\end{aligned}
$$

The last condition is implied by (17).

E.5.1 Proof of Lemma 5

Recall that, given aggregate output Y_t, if there are n firms in all industries at $t+1$, we have that

$$
Y_{t+1}=\iota_Y\Phi\left(\mathcal{F}_\pi,n_{t+1}\right)\left[\Theta\left(\mathcal{F}_\pi,n_{t+1}\right)\right]^{\varsigma_Y}Y_t^{\varepsilon_Y}
$$

Therefore

$$
\begin{aligned}
\iota_Y\ \Phi\left(\mathcal{F}_\pi,n\right)\ \left[\Theta\left(\mathcal{F}_\pi,n\right)\right]^{\varsigma_Y}\ \left[\underline{Y}_0\left(\mathcal{F}_\pi,n\right)\right]^{\varepsilon_Y} &= \frac{c_f}{\mathit{\Lambda}\left(\mathcal{F}_\pi,n\right)}\left[\Theta\left(\mathcal{F}_\pi,n\right)\right]^{\frac{\rho}{1-\rho}} \\
\iota_Y\ \Phi\left(\mathcal{F}_\pi,n\right)\ \left[\Theta\left(\mathcal{F}_\pi,n\right)\right]^{\varsigma_Y}\ \left[\overline{Y}_0\left(\mathcal{F}_\pi,n\right)\right]^{\varepsilon_Y} &= \frac{c_f}{\mathit{\Lambda}\left(\mathcal{F}_\pi,n+1\right)}\left[\Theta\left(\mathcal{F}_\pi,n\right)\right]^{\frac{\rho}{1-\rho}}
\end{aligned}
$$

which is equivalent to

$$
\begin{aligned}
\underline{Y}_0\left(\mathcal{F}_\pi,n\right) &:= \left\{\iota_Y^{-1}\frac{c_f}{\mathit{\Lambda}\left(\mathcal{F}_\pi,n\right)}\left[\Phi\left(\mathcal{F}_\pi,n\right)\right]^{-1}\left[\Theta\left(\mathcal{F}_\pi,n\right)\right]^{\frac{\rho}{1-\rho}-\varsigma_Y}\right\}^{\frac{1}{\varepsilon_Y}} \\
\overline{Y}_0\left(\mathcal{F}_\pi,n\right) &:= \left\{\iota_Y^{-1}\frac{c_f}{\mathit{\Lambda}\left(\mathcal{F}_\pi,n+1\right)}\left[\Phi\left(\mathcal{F}_\pi,n\right)\right]^{-1}\left[\Theta\left(\mathcal{F}_\pi,n\right)\right]^{\frac{\rho}{1-\rho}-\varsigma_Y}\right\}^{\frac{1}{\varepsilon_Y}}
\end{aligned}
$$

E.5.2 Proof of Lemma 7

Part I Recall that

$$
\Theta\left(\mathcal{F}_\pi,n_t\right):=\frac{n-(1-\rho)}{\sum_{k=1}^{n}\frac{1}{\pi^k}}
$$

We therefore have that for any type $k\leq n$

$$
\frac{\partial\Theta\left(\mathcal{F}_\pi,n_t\right)}{\partial\pi^k}>0
$$

As we have seen above, $\mathit{\Lambda}(\cdot)$ is decreasing in $\dfrac{n-(1-\rho)}{\sum_{k=1}^{n}\frac{1}{\pi^k}}$. Therefore, when π^k increases, and for any given $k<n$,

(i) $\Theta(\mathcal{F}_\pi, n_t)$ increases

(ii) $\mathit{\Lambda}(\mathcal{F}_\pi, n)$ and $\mathit{\Lambda}(\mathcal{F}_\pi, n+1)$ decrease

We therefore have that both

$$\begin{aligned}
\underline{Y}(\mathcal{F}_\pi, n) &:= \frac{c_f}{\mathit{\Lambda}(\mathcal{F}_\pi, n)}\left[\Theta(\mathcal{F}_\pi, n_t)\right]^{\frac{\rho}{1-\rho}} \\
\overline{Y}(\mathcal{F}_\pi, n) &:= \frac{c_f}{\mathit{\Lambda}(\mathcal{F}_\pi, n+1)}\left[\Theta(\mathcal{F}_\pi, n_t)\right]^{\frac{\rho}{1-\rho}}
\end{aligned}$$

increase.

Part II We want to have

$$\frac{\partial}{\partial \pi^k}\left[\frac{\underline{Y}_0(\mathcal{F}_\pi, n)}{\underline{Y}(\mathcal{F}_\pi, n)}\right] > 0$$

Note that we have

$$\begin{aligned}
\underline{Y}_0(\mathcal{F}_\pi, n) = &\left\{\iota_Y^{-1}\frac{c_f}{\mathit{\Lambda}(\mathcal{F}_\pi, n)}\left[\Phi(\mathcal{F}_\pi, n)\right]^{-1}\left[\Theta(\mathcal{F}_\pi, n)\right]^{\frac{\rho}{1-\rho}-\varsigma_Y}\right\}^{\frac{1}{\varepsilon_Y}} \\
= &\left\{\iota_Y^{-1}\left[\Phi(\mathcal{F}_\pi, n)\right]^{-1}\left[\Theta(\mathcal{F}_\pi, n)\right]^{-\varsigma_Y}\underline{Y}(\mathcal{F}_\pi, n)\right\}^{\frac{1}{\varepsilon_Y}} \\
\Rightarrow \frac{\underline{Y}_0(\mathcal{F}_\pi, n)}{\underline{Y}(\mathcal{F}_\pi, n)} = &\left\{\iota_Y^{-1}\left[\Phi(\mathcal{F}_\pi, n)\right]^{-1}\left[\Theta(\mathcal{F}_\pi, n)\right]^{-\varsigma_Y}\right\}^{\frac{1}{\varepsilon_Y}}\left[\underline{Y}(\mathcal{F}_\pi, n)\right]^{\frac{1}{\varepsilon_Y}-1}
\end{aligned}$$

Furthermore, recall that

1. $\underline{Y}(\mathcal{F}_\pi, n)$

$$\begin{aligned}
\underline{Y}(\mathcal{F}_\pi, n) = &\frac{c_f}{\mathit{\Lambda}(\mathcal{F}_\pi, n)}\left[\Theta(\mathcal{F}_\pi, n)\right]^{\frac{\rho}{1-\rho}} \\
\Rightarrow \frac{\partial \underline{Y}(\mathcal{F}_\pi, n)}{\partial \pi^k} = &\frac{\rho}{1-\rho}\frac{c_f}{\mathit{\Lambda}(\mathcal{F}_\pi, n)}\left[\Theta(\mathcal{F}_\pi, n)\right]^{\frac{\rho}{1-\rho}-1}\frac{\partial \Theta(\mathcal{F}_\pi, n)}{\partial \pi^k} - \frac{c_f}{\left[\mathit{\Lambda}(\mathcal{F}_\pi, n)\right]^2}\left[\Theta(\mathcal{F}_\pi, n)\right]^{\frac{\rho}{1-\rho}}\frac{\partial \mathit{\Lambda}(\mathcal{F}_\pi, n)}{\partial \pi^k} \\
= &\underline{Y}(\mathcal{F}_\pi, n)\left[\frac{\rho}{1-\rho}\left[\Theta(\mathcal{F}_\pi, n)\right]^{-1}\frac{\partial \Theta(\mathcal{F}_\pi, n)}{\partial \pi^k} - \left[\mathit{\Lambda}(\mathcal{F}_\pi, n)\right]^{-1}\frac{\partial \mathit{\Lambda}(\mathcal{F}_\pi, n)}{\partial \pi^k}\right]
\end{aligned}$$

2. $\Phi(\mathcal{F}_\pi, n)$

$$\begin{aligned}
\Phi(\mathcal{F}_\pi, n_t) = &(1-\rho)\left\{\sum_{k=1}^{n}\left[\frac{1}{\pi^k} - \Theta(\mathcal{F}_\pi, n_t)\left(\frac{1}{\pi^k}\right)^2\right]\right\}^{-1} \\
\Rightarrow \frac{\partial \Phi(\mathcal{F}_\pi, n)}{\partial \pi^k} = &(1-\rho)\frac{-\left[-\left(\frac{1}{\pi^k}\right)^2 + 2\Theta(\mathcal{F}_\pi, n_t)\left(\frac{1}{\pi^k}\right)^3 - \frac{\partial \Theta(\mathcal{F}_\pi, n)}{\partial \pi^k}\left(\frac{1}{\pi^k}\right)^2\right]}{\left\{\sum_{k=1}^{n}\left[\frac{1}{\pi^k} - \Theta(\mathcal{F}_\pi, n_t)\left(\frac{1}{\pi^k}\right)^2\right]\right\}^2} \\
= &\Phi(\mathcal{F}_\pi, n)\frac{\left(\frac{1}{\pi^k}\right)^2 - 2\Theta(\mathcal{F}_\pi, n_t)\left(\frac{1}{\pi^k}\right)^3 + \frac{\partial \Theta(\mathcal{F}_\pi, n)}{\partial \pi^k}\left(\frac{1}{\pi^k}\right)^2}{\sum_{k=1}^{n}\left[\frac{1}{\pi^k} - \Theta(\mathcal{F}_\pi, n_t)\left(\frac{1}{\pi^k}\right)^2\right]} \\
= &\frac{\left[\Phi(\mathcal{F}_\pi, n)\right]^2}{1-\rho}\left(\frac{1}{\pi^k}\right)^2\left\{1 - 2\Theta(\mathcal{F}_\pi, n_t)\frac{1}{\pi^k} + \frac{\partial \Theta(\mathcal{F}_\pi, n)}{\partial \pi^k}\right\}
\end{aligned}$$

3. $\Theta\left(\mathcal{F}_\pi, n\right)$

$$\begin{aligned}
\Theta\left(\mathcal{F}_\pi, n_t\right) &= \frac{n-(1-\rho)}{\sum_{k=1}^{n} \frac{1}{\pi^k}} \\
\Rightarrow \frac{\partial \Theta\left(\mathcal{F}_\pi, n\right)}{\partial \pi^k} &= \frac{-\left[n-(1-\rho)\right]\left[-\left(\frac{1}{\pi^k}\right)^2\right]}{\left[\sum_{k=1}^{n} \frac{1}{\pi^k}\right]^2} \\
&= \Theta\left(\mathcal{F}_\pi, n\right) \frac{\left(\frac{1}{\pi^k}\right)^2}{\sum_{k=1}^{n} \frac{1}{\pi^k}}
\end{aligned}$$

3. $\mathit{\Lambda}\left(\mathcal{F}_\pi, n\right)$

$$\mathit{\Lambda}\left(\mathcal{F}_\pi, \pi^j, n\right) = \frac{1}{1-\rho}\left[\frac{\sum_{k=1}^{n} \frac{1}{\pi^k}}{n-(1-\rho)} - \frac{1}{\pi^j}\right]\left[1 - \frac{n-(1-\rho)}{\sum_{k=1}^{n} \frac{1}{\pi^k}} \frac{1}{\pi^j}\right]\left[\frac{n-(1-\rho)}{\sum_{k=1}^{n} \frac{1}{\pi^k}}\right]^{\frac{1}{1-\rho}}$$

We have that

$$
\begin{aligned}
\mathit{\Lambda}\left(\mathcal{F}_\pi,\pi^j,n\right) =&\quad \frac{1}{1-\rho}\left[\frac{\sum_{k=1}^n \frac{1}{\pi^k}}{n-(1-\rho)}-\frac{1}{\pi^j}\right]\left[1-\frac{n-(1-\rho)}{\sum_{k=1}^n \frac{1}{\pi^k}}\frac{1}{\pi^j}\right]\left[\frac{n-(1-\rho)}{\sum_{k=1}^n \frac{1}{\pi^k}}\right]^{\frac{1}{1-\rho}} \\
\Rightarrow \frac{\partial \mathit{\Lambda}\left(\mathcal{F}_\pi,\pi^j,n\right)}{\partial \pi^k} =&\quad \mathit{\Lambda}\left(\mathcal{F}_\pi,\pi^j,n\right)\left\{\left[\frac{\sum_{k=1}^n \frac{1}{\pi^k}}{n-(1-\rho)}-\frac{1}{\pi^j}\right]^{-1}\frac{-\left(\frac{1}{\pi^k}\right)^2}{n-(1-\rho)}\right. \\
&+\left[1-\frac{n-(1-\rho)}{\sum_{k=1}^n \frac{1}{\pi^k}}\frac{1}{\pi^j}\right]^{-1}\frac{-\left\{-\left[n-(1-\rho)\right]\left[-\left(\frac{1}{\pi^k}\right)^2\right]\right\}}{\left[\sum_{k=1}^n \frac{1}{\pi^k}\right]^2}\frac{1}{\pi^j} \\
&\left.+\frac{1}{1-\rho}\left[\frac{n-(1-\rho)}{\sum_{k=1}^n \frac{1}{\pi^k}}\right]^{-1}\frac{-\left[n-(1-\rho)\right]\left[-\left(\frac{1}{\pi^k}\right)^2\right]}{\left[\sum_{k=1}^n \frac{1}{\pi^k}\right]^2}\right\} \\
=&\quad \mathit{\Lambda}\left(\mathcal{F}_\pi,n\right)\left(\frac{1}{\pi^k}\right)^2\left\{\frac{n-(1-\rho)}{\left[\sum_{k=1}^n \frac{1}{\pi^k}\right]^2}\left\{\frac{1}{1-\rho}\left[\frac{n-(1-\rho)}{\sum_{k=1}^n \frac{1}{\pi^k}}\right]^{-1}-\left[1-\frac{n-(1-\rho)}{\sum_{k=1}^n \frac{1}{\pi^k}}\frac{1}{\pi^j}\right]^{-1}\frac{1}{\pi^j}\right\}\right. \\
&\left.-\left[\sum_{k=1}^n \frac{1}{\pi^k}-\frac{n-(1-\rho)}{\pi^j}\right]^{-1}\right\} \\
=&\quad \mathit{\Lambda}\left(\mathcal{F}_\pi,n\right)\left(\frac{1}{\pi^k}\right)^2\left\{\frac{1}{1-\rho}\left(\sum_{k=1}^n \frac{1}{\pi^k}\right)^{-1}-\left[\frac{\left[\sum_{k=1}^n \frac{1}{\pi^k}\right]^2}{n-(1-\rho)}-\frac{\sum_{k=1}^n \frac{1}{\pi^k}}{\pi^j}\right]^{-1}\frac{1}{\pi^j}\right. \\
&\left.-\left[\sum_{k=1}^n \frac{1}{\pi^k}-\frac{n-(1-\rho)}{\pi^j}\right]^{-1}\right\}
\end{aligned}
$$

We hence have that

$$\begin{aligned}\frac{\partial}{\partial \pi^k}\left[\frac{\underline{Y}_0(\mathcal{F}_\pi,n)}{\underline{Y}(\mathcal{F}_\pi,n)}\right] &= \frac{\underline{Y}_0(\mathcal{F}_\pi,n)}{\underline{Y}(\mathcal{F}_\pi,n)}\left\{-\frac{1}{\varepsilon_Y}\left[\Phi(\mathcal{F}_\pi,n)\right]^{-1}\frac{\partial \Phi(\mathcal{F}_\pi,n)}{\partial \pi^k} - \frac{\varsigma_Y}{\varepsilon_Y}\left[\Theta(\mathcal{F}_\pi,n)\right]^{-1}\frac{\partial \Theta(\mathcal{F}_\pi,n)}{\partial \pi^k} + \left(\frac{1}{\varepsilon_Y}-1\right)\left[\underline{Y}(\mathcal{F}_\pi,n)\right]^{-1}\frac{\partial \underline{Y}(\mathcal{F}}{\partial \pi}\right. \\ &= \frac{\underline{Y}_0(\mathcal{F}_\pi,n)}{\underline{Y}(\mathcal{F}_\pi,n)}\frac{1}{\varepsilon_Y}\left\{-\left[\Phi(\mathcal{F}_\pi,n)\right]^{-1}\frac{\partial \Phi(\mathcal{F}_\pi,n)}{\partial \pi^k} - \varsigma_Y\left[\Theta(\mathcal{F}_\pi,n)\right]^{-1}\frac{\partial \Theta(\mathcal{F}_\pi,n)}{\partial \pi^k} + (1-\varepsilon_Y)\left[\underline{Y}(\mathcal{F}_\pi,n)\right]^{-1}\frac{\partial \underline{Y}(\mathcal{F}_\pi,}{\partial \pi^k}\right.\end{aligned}$$

$$\begin{aligned}&\frac{\partial}{\partial \pi^k}\left[\frac{\underline{Y}_0(\mathcal{F}_\pi,n)}{\underline{Y}(\mathcal{F}_\pi,n)}\right] > 0 \\ \Leftrightarrow\ & -\left[\Phi(\mathcal{F}_\pi,n)\right]^{-1}\frac{\partial \Phi(\mathcal{F}_\pi,n)}{\partial \pi^k} - \varsigma_Y\left[\Theta(\mathcal{F}_\pi,n)\right]^{-1}\frac{\partial \Theta(\mathcal{F}_\pi,n)}{\partial \pi^k} + (1-\varepsilon_Y)\left[\underline{Y}(\mathcal{F}_\pi,n)\right]^{-1}\frac{\partial \underline{Y}(\mathcal{F}_\pi,n)}{\partial \pi^k} > 0 \\ \Leftrightarrow\ & -\frac{\Phi(\mathcal{F}_\pi,n)}{1-\rho}\left(\frac{1}{\pi^k}\right)^2\left\{1-2\Theta(\mathcal{F}_\pi,n_t)\frac{1}{\pi^k}+\frac{\partial \Theta(\mathcal{F}_\pi,n)}{\partial \pi^k}\right\} \\ & -\varsigma_Y\frac{\left(\frac{1}{\pi^k}\right)^2}{\sum_{k=1}^{n}\frac{1}{\pi^k}} + (1-\varepsilon_Y)\underbrace{\left[\frac{\rho}{1-\rho}\left[\Theta(\mathcal{F}_\pi,n)\right]^{-1}\frac{\partial \Theta(\mathcal{F}_\pi,n)}{\partial \pi^k} - \left[\Lambda(\mathcal{F}_\pi,n)\right]^{-1}\frac{\partial \Lambda(\mathcal{F}_\pi,n)}{\partial \pi^k}\right]}_{>0} > 0\end{aligned}$$

Recall that ς_Y and ε_Y are both increasing in the labor supply elasticity $\frac{1}{\nu}$. Therefore, the above condition will typically be satisfied when the elasticity $\frac{1}{\nu}$ is low. To provide further intuition, we evaluate the above condition in the special case of a duopoly ($n=1$) and when the two firms are equally productive ($\pi^1=\pi^2=1$).

$$\begin{aligned}& -\frac{1}{1-\rho}\left(1-2\frac{1+\rho}{2}+\frac{1+\rho}{2}\frac{1}{2}\right) \\ & -\frac{\varsigma_Y}{2}+(1-\varepsilon_Y)\left[\frac{\rho}{2(1-\rho)}-\left\{\frac{1}{2(1-\rho)}-\frac{1+\rho}{2(1-\rho)}-\frac{1}{1-\rho}\right\}\right] > 0 \\ \Leftrightarrow\ & -\frac{1}{1-\rho}\left(-\rho+\frac{1+\rho}{4}\right)-\frac{\varsigma_Y}{2}+(1-\varepsilon_Y)\left[\frac{\rho}{2(1-\rho)}-\frac{1}{2(1-\rho)}+\frac{1+\rho}{2(1-\rho)}+\frac{1}{1-\rho}\right] \\ \Leftrightarrow\ & \frac{1}{1-\rho}\left(\rho-\frac{1+\rho}{4}\right)-\frac{\varsigma_Y}{2}+(1-\varepsilon_Y)\left[\frac{2\rho}{2(1-\rho)}+\frac{1}{1-\rho}\right] > 0 \\ \Leftrightarrow\ & \frac{1}{1-\rho}\left(\rho-\frac{1+\rho}{4}\right)-\frac{\varsigma_Y}{2}+(1-\varepsilon_Y)\frac{1+\rho}{1-\rho} > 0 \\ \Leftrightarrow\ & \frac{\rho}{1-\rho}+\frac{1+\rho}{1-\rho}\left[(1-\varepsilon_Y)-\frac{1}{4}\right]-\frac{\varsigma_Y}{2} > 0\end{aligned}$$

$$\begin{aligned}\varepsilon_Y &= \tfrac{\alpha(1+\nu)}{\nu+\alpha+\gamma\nu(1-\alpha)} \\ \zeta &= \tfrac{(1-\alpha)+\gamma(1+\nu\alpha)}{\nu+\alpha+\gamma\nu(1-\alpha)}\end{aligned}$$

E.6 Asymmetric Equilibrium

Suppose that

$$\overline{Y}_0\left(\mathcal{F}_\pi, n\right) < Y_t < \underline{Y}_0\left(\mathcal{F}_\pi, n+1\right)$$

In such case there will be an asymmetric equilibrium at time $t+1$: some industries will contain n firms, whereas some industries will contain $n+1$ firms. The fraction of industries with $n+1$ will be pinned down by a zero profit condition for the marginal entrant in an industry with $n+1$ firms

$$\Lambda\left(\mathcal{F}_\pi, \pi^{n+1}, n+1\right)\Theta_t^{-\frac{\rho}{1-\rho}}Y_t = c_f$$

We also have that

$$\begin{aligned}
y_n &= \left[\frac{\sum_{k=1}^{n}\frac{1}{\pi^k}}{n-(1-\rho)}\Theta\right]^{-\frac{1}{1-\rho}} Y_t \\
y_{n+1} &= \left[\frac{\sum_{k=1}^{n+1}\frac{1}{\pi^k}}{n+1-(1-\rho)}\Theta\right]^{-\frac{1}{1-\rho}} Y_t \\
\Rightarrow \quad y_{n+1} &= \underbrace{\left[\frac{n-(1-\rho)}{n+1-(1-\rho)}\frac{\sum_{k=1}^{n+1}\frac{1}{\pi^k}}{\sum_{k=1}^{n}\frac{1}{\pi^k}}\right]^{-\frac{1}{1-\rho}}}_{:=\Upsilon(n)} y_n
\end{aligned}$$

We can hence write

$$\begin{aligned}
& Y = \left\{\upsilon + (1-\upsilon)\left[\Upsilon(n)\right]^\rho\right\}^{\frac{1}{\rho}} y_n \\
\Leftrightarrow \quad & Y = \left\{\upsilon + (1-\upsilon)\left[\Upsilon(n)\right]^\rho\right\}^{\frac{1}{\rho}} \left[\frac{\sum_{k=1}^{n}\frac{1}{\pi^k}}{n-(1-\rho)}\Theta_t\right]^{-\frac{1}{1-\rho}} Y \\
\Leftrightarrow \quad & 1 = \left\{\upsilon + (1-\upsilon)\left[\Upsilon(n)\right]^\rho\right\}^{\frac{1-\rho}{\rho}} \left[\frac{\sum_{k=1}^{n}\frac{1}{\pi^k}}{n-(1-\rho)}\Theta\right]^{-1} \\
\Leftrightarrow \quad & \Theta = \left\{\upsilon + (1-\upsilon)\left[\Upsilon(n)\right]^\rho\right\}^{\frac{1-\rho}{\rho}} \left[\frac{\sum_{k=1}^{n}\frac{1}{\pi^k}}{n-(1-\rho)}\right]^{-1}
\end{aligned}$$

Let $\tilde{\text{is}}_n^j$ denote the *aggregate* input share of all types π^j producing in an industry with n firms

$$\begin{aligned} \tilde{\text{is}}_n^j &:= \frac{\int_0^{\upsilon} l_{it}^j \, di}{L_t} = \frac{\int_0^{\upsilon} k_{it}^j \, di}{K_t} \\ \tilde{\text{is}}_{n+1}^j &:= \frac{\int_{\upsilon}^1 l_{it}^j \, di}{L_t} = \frac{\int_{\upsilon}^1 k_{it}^j \, di}{K_t} \end{aligned}$$

Note that firm j produces $\frac{s_n^j}{s_n^1}$ as much output as firm 1 of the same industry and uses $\frac{s_n^j}{s_n^1}\frac{\pi^1}{\pi^j}$ as many inputs. Also note that a type π^j in an industry with $n+1$ firms uses $\frac{s_{n+1}^j}{s_n^j}\Upsilon(n)$ as much inputs as the same type in an industry with n firms. We therefore have that

$$\begin{aligned} & \upsilon \textstyle\sum_{k=1}^n \tilde{\text{is}}_n^k + (1-\upsilon)\sum_{k=1}^{n+1} \tilde{\text{is}}_{n+1}^k = 1 \\ \Leftrightarrow\ & \upsilon \textstyle\sum_{k=1}^n \tilde{\text{is}}_n^1 \frac{s_n^k}{s_n^1}\frac{\pi^1}{\pi^k} + (1-\upsilon)\sum_{k=1}^{n+1} \tilde{\text{is}}_{n+1}^1 \frac{s_{n+1}^k}{s_{n+1}^1}\frac{\pi^1}{\pi^k} = 1 \\ \Leftrightarrow\ & \upsilon \tilde{\text{is}}_n^1 \frac{\pi^1}{s_n^1} \textstyle\sum_{k=1}^n \frac{s_n^k}{\pi^k} + (1-\upsilon)\tilde{\text{is}}_{n+1}^1 \frac{\pi^1}{s_{n+1}^1} \sum_{k=1}^{n+1} \frac{s_{n+1}^k}{\pi^k} = 1 \\ \Leftrightarrow\ & \upsilon \tilde{\text{is}}_n^1 \frac{\pi^1}{s_n^1} \textstyle\sum_{k=1}^n \frac{s_n^k}{\pi^k} + (1-\upsilon)\left[\frac{s_{n+1}^1}{s_n^1}\Upsilon(n)\,\tilde{\text{is}}_n^1\right] \frac{\pi^1}{s_{n+1}^1} \sum_{k=1}^{n+1} \frac{s_{n+1}^k}{\pi^k} = 1 \\ \Leftrightarrow\ & \tilde{\text{is}}_n^1 \frac{\pi^1}{s_n^1}\left[\upsilon \textstyle\sum_{k=1}^n \frac{s_n^k}{\pi^k} + (1-\upsilon)\Upsilon(n)\sum_{k=1}^n \frac{s_{n+1}^k}{\pi^k}\right] = 1 \\ \Leftrightarrow\ & \tilde{\text{is}}_n^1 = \frac{s_n^1}{\pi^1}\left[\upsilon \textstyle\sum_{k=1}^n \frac{s_n^k}{\pi^k} + (1-\upsilon)\Upsilon(n)\sum_{k=1}^n \frac{s_{n+1}^k}{\pi^k}\right]^{-1} \end{aligned}$$

Note that we can write aggregate output as

$$\begin{aligned} & Y_t = \left[\upsilon y_n^{\rho} + (1-\upsilon) y_{n+1}^{\rho}\right]^{\frac{1}{\rho}} \\ \Leftrightarrow\ & Y_t = \{\upsilon + (1-\upsilon)[\Upsilon(n)]^{\rho}\}^{\frac{1}{\rho}} y_n \\ \Leftrightarrow\ & Y_t = \{\upsilon + (1-\upsilon)[\Upsilon(n)]^{\rho}\}^{\frac{1}{\rho}} \textstyle\sum_{k=1}^n y_n^k \\ \Leftrightarrow\ & Y_t = \{\upsilon + (1-\upsilon)[\Upsilon(n)]^{\rho}\}^{\frac{1}{\rho}} \textstyle\sum_{k=1}^n \pi^k \left(\tilde{\text{is}}_n^k L\right)^{1-\alpha}\left(\tilde{\text{is}}_n^k K\right)^{\alpha} \\ \Leftrightarrow\ & Y_t = \{\upsilon + (1-\upsilon)[\Upsilon(n)]^{\rho}\}^{\frac{1}{\rho}} \left(\textstyle\sum_{k=1}^n \pi^k \tilde{\text{is}}_n^k\right) L^{1-\alpha}K^{\alpha} \\ \Leftrightarrow\ & Y_t = \{\upsilon + (1-\upsilon)[\Upsilon(n)]^{\rho}\}^{\frac{1}{\rho}} \left(\textstyle\sum_{k=1}^n \pi^k \frac{s_n^k}{s_n^1}\frac{\pi^1}{\pi^k}\tilde{\text{is}}_n^1\right) L^{1-\alpha}K^{\alpha} \\ \Leftrightarrow\ & Y_t = \{\upsilon + (1-\upsilon)[\Upsilon(n)]^{\rho}\}^{\frac{1}{\rho}} \left(\frac{\pi^1}{s_n^1}\tilde{\text{is}}_n^1 \underbrace{\sum_{k=1}^n s_n^k}_{=1}\right) L^{1-\alpha}K^{\alpha} \\ \Leftrightarrow\ & Y_t = \underbrace{\{\upsilon + (1-\upsilon)[\Upsilon(n)]^{\rho}\}^{\frac{1}{\rho}} \left[\upsilon \sum_{k=1}^n \frac{s_n^k}{\pi^k} + (1-\upsilon)\Upsilon(n)\sum_{k=1}^n \frac{s_{n+1}^k}{\pi^k}\right]^{-1}}_{:=\Phi(n,\upsilon)} L^{1-\alpha}K^{\alpha} \\ \Leftrightarrow\ & Y_t = \Phi(n,\upsilon)\, L^{1-\alpha}K^{\alpha} \end{aligned}$$

Let υ be the fraction of industries with n firms. The aggregate equilibrium is pinned down by the following four equations

$$Y_{t+1} = \iota_Y \Phi_{t+1} \Theta_{t+1}^{\varsigma_Y} Y_t^{\varepsilon_Y}$$
$$\Lambda\left(\mathcal{F}_\pi, \pi^{n+1}, n+1\right) \Theta_{t+1}^{-\frac{\rho}{1-\rho}} Y_{t+1} = c_f$$
$$\Theta_{t+1} = \{\upsilon + (1-\upsilon)\left[\Upsilon(n)\right]^\rho\}^{\frac{1-\rho}{\rho}} \left[\frac{\sum_{k=1}^{n} \frac{1}{\pi^k}}{n-(1-\rho)}\right]^{-1}$$
$$\Phi_{t+1} = \{\upsilon + (1-\upsilon)\left[\Upsilon(n)\right]^\rho\}^{\frac{1}{\rho}} \left[\upsilon \sum_{k=1}^{n} \frac{s_n^k}{\pi^k} + (1-\upsilon)\Upsilon(n) \sum_{k=1}^{n} \frac{s_{n+1}^k}{\pi^k}\right]^{-1}$$

The market shares s_n^k and s_{n+1}^k are defined in Appendix E.1.2.

E.6.1 Transition to Full Monopoly

Let θ be the fraction of industries with a monopoly. We have

$$\begin{aligned} & Y_t = \left[\theta y_1^\rho + (1-\theta)\, 0\right]^{\frac{1}{\rho}} \\ \Leftrightarrow \quad & Y_t = \theta^{\frac{1}{\rho}} y_1 \end{aligned}$$

The equilibrium is defined by

$$\begin{aligned} & Y_{t+1} = \iota_Y \Phi_{t+1} \Theta_{t+1}^{\varsigma_Y} Y_t^{\varepsilon_Y} \\ & \Lambda\left(\mathcal{F}_\pi, \pi^1, 1\right) \Theta_{t+1}^{-\frac{\rho}{1-\rho}} Y_{t+1} = c_f \\ & \Theta_{t+1} = \theta^{\frac{1-\rho}{\rho}} \rho \pi^1 \\ & \Phi_{t+1} = \theta^{\frac{1-\rho}{\rho}} \pi^1 \end{aligned}$$

We hence have that

$$\begin{aligned} & \Lambda\left(\mathcal{F}_\pi, \pi^1, 1\right) \left[\theta_{t+1}^{\frac{1-\rho}{\rho}} \left(\frac{1}{\rho\pi^1}\right)^{-1}\right]^{-\frac{\rho}{1-\rho}} Y_{t+1} = c_f \\ \Leftrightarrow \quad & \Lambda\left(\mathcal{F}_\pi, \pi^1, 1\right) \theta_{t+1}^{-1} \left(\frac{1}{\rho\pi^1}\right)^{\frac{\rho}{1-\rho}} Y_{t+1} = c_f \\ \Leftrightarrow \quad & \theta_{t+1} = \Lambda\left(\mathcal{F}_\pi, \pi^1, 1\right) \frac{Y_{t+1}}{c_f} \left(\frac{1}{\rho\pi^1}\right)^{\frac{\rho}{1-\rho}} \\ \Leftrightarrow \quad & \theta_{t+1} = (1-\rho) \left(\rho\pi^1\right)^{\frac{\rho}{1-\rho}} \frac{Y_{t+1}}{c_f} \left(\frac{1}{\rho\pi^1}\right)^{\frac{\rho}{1-\rho}} \\ \Leftrightarrow \quad & \theta_{t+1} = (1-\rho) \frac{Y_{t+1}}{c_f} \end{aligned}$$

So that

$$\begin{aligned} Y_{t+1} = &\quad \iota_Y \theta^{\frac{1-\rho}{\rho}} \pi^1 \left(\theta^{\frac{1-\rho}{\rho}} \rho \pi^1\right)^{\varsigma_Y} Y_t^{\varepsilon_Y} \\ \Leftrightarrow Y_{t+1} = &\quad \iota_Y \rho^{\varsigma_Y} \left(\theta^{\frac{1-\rho}{\rho}} \pi^1\right)^{1+\varsigma_Y} Y_t^{\varepsilon_Y} \\ \Leftrightarrow Y_{t+1} = &\quad \iota_Y \rho^{\varsigma_Y} \left[\left(\frac{1-\rho}{c_f} Y_{t+1}\right)^{\frac{1-\rho}{\rho}} \pi^1\right]^{1+\varsigma_Y} Y_t^{\varepsilon_Y} \\ \Leftrightarrow Y_{t+1}^{1-\frac{1-\rho}{\rho}(1+\varsigma_Y)} = &\quad \iota_Y \rho^{\varsigma_Y} \left[\left(\frac{1-\rho}{c_f}\right)^{\frac{1-\rho}{\rho}} \pi^1\right]^{1+\varsigma_Y} Y_t^{\varepsilon_Y} \end{aligned}$$

E.7 Multiple Equilibria

E.7.1 Proof of Lemma 6

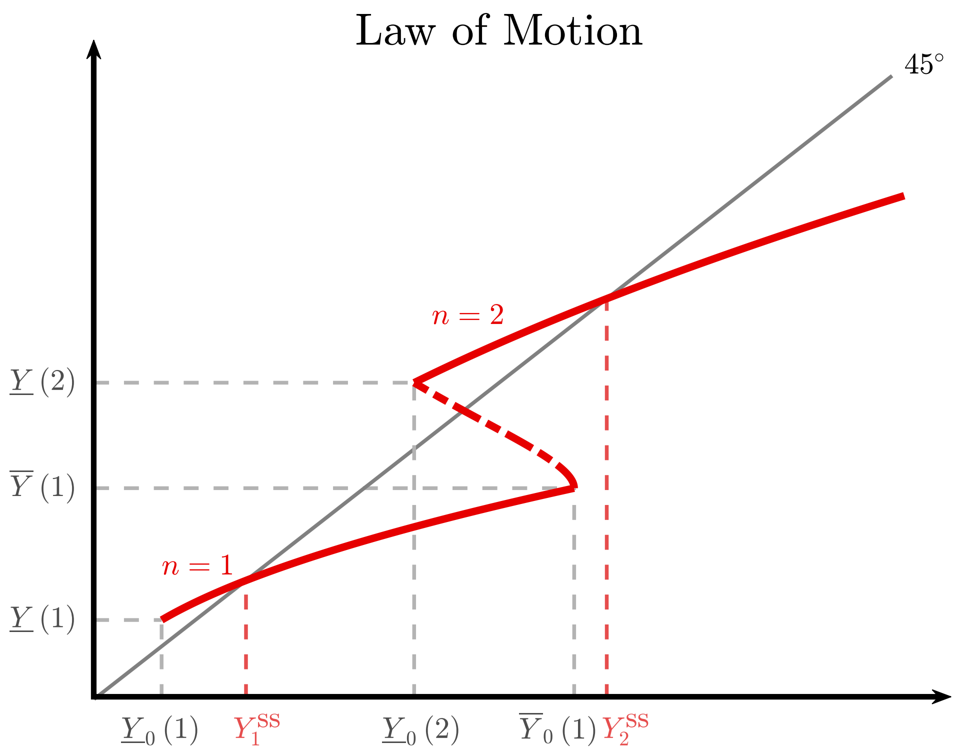
Symmetric equilibria with n and $n+1$ firms per industry are simultaneously possible if

$$
\begin{aligned}
& \underline{Y}_0\left(\mathcal{F}_\pi, n+1\right) < \overline{Y}_0\left(\mathcal{F}_\pi, n\right) \\
\Leftrightarrow \quad & \frac{\left[\Phi\left(\mathcal{F}_\pi, n+1\right)\right]^{-1}\left[\Theta\left(\mathcal{F}_\pi, n+1\right)\right]^{\frac{\rho}{1-\rho}-\varsigma_Y}}{\Lambda\left(\mathcal{F}_\pi, n+1\right)} < \frac{\left[\Phi\left(\mathcal{F}_\pi, n\right)\right]^{-1}\left[\Theta\left(\mathcal{F}_\pi, n\right)\right]^{\frac{\rho}{1-\rho}-\varsigma_Y}}{\Lambda\left(\mathcal{F}_\pi, n+1\right)} \\
\Leftrightarrow \quad & \left[\Phi\left(\mathcal{F}_\pi, n+1\right)\right]^{-1}\left[\Theta\left(\mathcal{F}_\pi, n+1\right)\right]^{\frac{\rho}{1-\rho}-\varsigma_Y} < \left[\Phi\left(\mathcal{F}_\pi, n\right)\right]^{-1}\left[\Theta\left(\mathcal{F}_\pi, n\right)\right]^{\frac{\rho}{1-\rho}-\varsigma_Y} \\
\Leftrightarrow \quad & \frac{\Phi\left(\mathcal{F}_\pi, n\right)}{\Phi\left(\mathcal{F}_\pi, n+1\right)} < \left[\frac{\Theta\left(\mathcal{F}_\pi, n\right)}{\Theta\left(\mathcal{F}_\pi, n+1\right)}\right]^{\frac{\rho}{1-\rho}-\varsigma_Y}
\end{aligned}
$$

Recall that we always have that $\frac{\Theta\left(\mathcal{F}_\pi, n\right)}{\Theta\left(\mathcal{F}_\pi, n+1\right)} < 1$. If there are no productivity differences across firms, we further have that $\frac{\Phi\left(\mathcal{F}_\pi, n\right)}{\Phi\left(\mathcal{F}_\pi, n+1\right)} = 1$. Therefore, for the above condition to be satisfied we need that

$$
\begin{aligned}
& \frac{\rho}{1-\rho} - \underbrace{\frac{(1-\alpha)+\gamma(1+\nu\alpha)}{\nu+\alpha+\gamma\nu(1-\alpha)}}_{=\varsigma_Y} < 0 \\
\Leftrightarrow \quad & \rho\left[\nu+\alpha+\gamma\nu(1-\alpha)\right] < (1-\rho)\left[(1-\alpha)+\gamma(1+\nu\alpha)\right] \\
\Leftrightarrow \quad & \rho\left[\nu+\gamma\nu+1+\gamma\right] < (1-\alpha)+\gamma(1+\nu\alpha) \\
\Leftrightarrow \quad & \rho(1+\gamma)(1+\nu) < (1+\gamma)+\alpha(\nu-1) \\
\Leftrightarrow \quad & \rho < \frac{(1+\gamma)+\alpha(\nu-1)}{(1+\gamma)(1+\nu)}
\end{aligned}
$$

Figure 24 shows a law of motion that features multiple equilibria and multiple steady-states.



Parameters: $\rho = 0.5$, $\alpha = 0.4$, $\nu = 0.5$, $\eta = 0.2$, $\gamma = 1$ and $c_f = 0.075$

Figure 24: Law of Motion: No productivity differences

E.8 Static Multiplicity (new proof)

E.8.1 Symmetric Equilibria

When n_i firms produce in a given industry i, and given aggregate output Y_t and factor costs Θ_t, type π^j makes production profits

$$\Pi\left(\pi^j, n_i, \Theta_t, Y_t\right) = \Lambda\left(\pi^j, n_i\right)\Theta_t^{-\frac{\rho}{1-\rho}} Y_t$$

Furthermore, given aggregate capital K_t, if there are n_t firms in all industries, we have that

$$Y_t = \Phi\left(\mathcal{F}_\pi, n_t\right)\left[\frac{(1-\alpha)\,\Theta\left(\mathcal{F}_\pi, n_t\right)}{\eta}\right]^{\frac{1-\alpha}{\nu+\alpha}} K_t^{\alpha\frac{1+\nu}{\nu+\alpha}}$$

A symmetric equilibrium with n firms in all industries is therefore possible provided that

$$\Lambda\left(n, \pi^n\right)\left[\Theta\left(\mathcal{F}_\pi, n\right)\right]^{\frac{1-\alpha}{\nu+\alpha}-\frac{\rho}{1-\rho}}\Phi\left(\mathcal{F}_\pi, n\right)\left(\frac{1-\alpha}{\eta}\right)^{\frac{1-\alpha}{\nu+\alpha}} K_t^{\alpha\frac{1+\nu}{\nu+\alpha}} \geq c_f$$
$$\Lambda\left(n, \pi^{n+1}\right)\left[\Theta\left(\mathcal{F}_\pi, n\right)\right]^{\frac{1-\alpha}{\nu+\alpha}-\frac{\rho}{1-\rho}}\Phi\left(\mathcal{F}_\pi, n\right)\left(\frac{1-\alpha}{\eta}\right)^{\frac{1-\alpha}{\nu+\alpha}} K_t^{\alpha\frac{1+\nu}{\nu+\alpha}} \leq c_f$$

We can therefore define thresholds

$$\underline{K}\left(\mathcal{F}_{\pi},n\right):=\left\{\frac{c_f}{\Lambda\left(n,\pi^{n}\right)}\left(\frac{\eta}{1-\alpha}\right)^{\frac{1-\alpha}{\nu+\alpha}}\left[\Phi\left(\mathcal{F}_{\pi},n\right)\right]^{-1}\left[\Theta\left(\mathcal{F}_{\pi},n\right)\right]^{\frac{\rho}{1-\rho}-\frac{1-\alpha}{\nu+\alpha}}\right\}^{\frac{\nu+\alpha}{\alpha(1+\nu)}}$$

$$\overline{K}\left(\mathcal{F}_{\pi},n\right):=\left\{\frac{c_f}{\Lambda\left(n+1,\pi^{n+1}\right)}\left(\frac{\eta}{1-\alpha}\right)^{\frac{1-\alpha}{\nu+\alpha}}\left[\Phi\left(\mathcal{F}_{\pi},n\right)\right]^{-1}\left[\Theta\left(\mathcal{F}_{\pi},n\right)\right]^{\frac{\rho}{1-\rho}-\frac{1-\alpha}{\nu+\alpha}}\right\}^{\frac{\nu+\alpha}{\alpha(1+\nu)}}$$

$$\Lambda\left(\pi^{j},n\right)=\frac{1}{1-\rho}\left[1-\frac{n-(1-\rho)}{\sum_{k=1}^{n}\frac{1}{\pi^{k}}}\frac{1}{\pi^{j}}\right]^{2}\left[\frac{n-(1-\rho)}{\sum_{k=1}^{n}\frac{1}{\pi^{k}}}\right]^{\frac{\rho}{1-\rho}}$$

$$\Theta\left(\mathcal{F}_{\pi},n_t\right):=\frac{n-(1-\rho)}{\sum_{k=1}^{n}\frac{1}{\pi^{k}}}$$

$$\Phi\left(n_t\right)=\left\{\sum_{k=1}^{n}\frac{1}{\pi^{k}}\frac{1}{1-\rho}\left[1-\frac{n-(1-\rho)}{\sum_{h=1}^{n}\frac{1}{\pi^{h}}}\frac{1}{\pi^{k}}\right]\right\}^{-1}$$

E.8.2 Static Multiplicity

Suppose that

$$\underline{K}\left(\mathcal{F}_{\pi},n\right)\leq K_t\leq\overline{K}\left(\mathcal{F}_{\pi},n\right)$$

so that a symmetric equilibrium with n firms in every industry is possible. A symmetric equilibrium with $n+1$ firms will also be possible provided that

$$\begin{aligned}
&\underline{K}\left(\mathcal{F}_{\pi},n+1\right)<\overline{K}\left(\mathcal{F}_{\pi},n\right)\\
\Leftrightarrow\quad&\frac{c_f}{\Lambda\left(n+1,\pi^{n+1}\right)}\left[\Phi\left(\mathcal{F}_{\pi},n+1\right)\right]^{-1}\left[\Theta\left(\mathcal{F}_{\pi},n+1\right)\right]^{\frac{\rho}{1-\rho}-\frac{1-\alpha}{\nu+\alpha}}<\frac{c_f}{\Lambda\left(n+1,\pi^{n+1}\right)}\left[\Phi\left(\mathcal{F}_{\pi},n\right)\right]^{-1}\left[\Theta\left(\mathcal{F}_{\pi},n\right)\right]^{\frac{\rho}{1-\rho}-\frac{1-\alpha}{\nu+\alpha}}\\
\Leftrightarrow\quad&\frac{\Phi\left(\mathcal{F}_{\pi},n\right)}{\Phi\left(\mathcal{F}_{\pi},n+1\right)}<\left[\frac{\Theta\left(\mathcal{F}_{\pi},n\right)}{\Theta\left(\mathcal{F}_{\pi},n+1\right)}\right]^{\frac{\rho}{1-\rho}-\frac{1-\alpha}{\nu+\alpha}}
\end{aligned}$$

Therefore, when there are no productivity differences, the condition becomes

$$\begin{aligned}
&\left[\frac{\Theta\left(\mathcal{F}_{\pi},n\right)}{\Theta\left(\mathcal{F}_{\pi},n+1\right)}\right]^{\frac{\rho}{1-\rho}-\frac{1-\alpha}{\nu+\alpha}}>1\\
\Leftrightarrow\quad&\frac{\rho}{1-\rho}-\frac{1-\alpha}{\nu+\alpha}<0\\
\Leftrightarrow\quad&\frac{\rho}{1-\rho}<\frac{1-\alpha}{\nu+\alpha}\\
\Leftrightarrow\quad&\rho<\frac{1-\alpha}{\nu+\alpha}\left(1-\rho\right)\\
\Leftrightarrow\quad&\rho\left(\nu+\alpha\right)<\left(1-\alpha\right)\left(1-\rho\right)\\
\Leftrightarrow\quad&\rho<\frac{1-\alpha}{\nu+2\alpha+1}
\end{aligned}$$