The frontier of Social Impact Finance: Theory and two case studies

Leonardo Becchetti[1], Fabio Pisani[1], Francesco Salustri[2], and Lorenzo Semplici[3]

[1]*Department of Economics and Finance, University of Rome 'Tor Vergata'*
[2]*Nuffield Department of Population Health, University of Oxford*
[3]*Department of Economics, Lumsa*

· This draft: 15 June 2019 ·
· Provisional and incomplete – Please do not circulate ·

Abstract

Social impact bonds (SIBs) are a novel and innovative form of public-private partnership financing social services performed by a best-practice selected non-governmental third entity. In our paper we outline a SIB theoretical model identifying government and private investors' participation constraints and the conflicts of interests that may arise among the different actors involved in presence of asymmetric information. We apply our theoretical model to two investment cases concerning contrast to jail recidivism and health budget project. We show conditions for viability of the SIB scheme in both cases under reasonable parametric conditions, provide sensitivity analysis on crucial parameters, and calculate participants' payoffs under different assumptions.

JEL numbers: G23 (financial instruments); I31 (General welfare, wellbeing).

Keywords: social impact finance, jail recidivism, health budget.

1 Introduction

Social impact bonds (SIBs) are innovative financial instruments that have recently drawn increasing attention from policymakers and researchers. They are financial mechanisms aimed at attracting private capital to finance the provision of a social service from a high-quality organisation, which is expected to reduce costs for the commissioners (usually public administrations) in service provision. SIBs are also called pay-for-success or pay-for-performance bonds (OECD, 2016; Gustaffson-Wright et al., 2015) since they incorporate a bet: if the performance of the social service provider is above a given threshold agreed among the counterparts, part of the costs saved by the commissioner become profits of the private investors financing the initiative.

The structure of SIBs is complex and involves several actors: a private financial intermediary issuing the bonds, private investors buying them, the commissioner (most often a government or a local administration), the social service provider, the beneficiaries of the social service, and the independent validator who ascertains whether the project effect attributable to the service provider overcomes an ex ante fixed performance threshold.

Given the above mentioned features, SIBs have great potential. First, SIBs leverage private financial resources to invest in the provision of social services by rolling over them the risk usually run by the commissioner. Second, SIBs stimulate a culture of quality and innovation in the provision of social services. In fact, within the SIB's scheme the financial intermediary retains a strong interest to select the highest quality provider to ensure the success of the operation. This implies that successful SIBs may be win-win operations as they can potentially generate permanent reductions of government budget and, at the same time, higher quality of public services (OECD, 2016).

In spite of this promising potential, some caveats related to the articulated SIB structure cannot be neglected. First, SIBs can work for a limited number of social activities, that is, those activities where the commissioner has a positive economic benefit and both the commissioner economic benefit and the service provider performance are clearly identifiable and measurable (i.e., they can be agreed upon by the counterparts). As such, SIBs have been so far used mainly in projects against jail recidivism[1] or school abandonment that configure clear costs for the public administration and potential benefits if the quality of service provision improves. However, SIBs can also find applications in job training, health care and prevention campaigns, provision of disability services, and foster care (OECD, 2016). It must be as well noted that, in principle, the boundaries of the viability of a SIB operation can be broadened in two directions. First, by considering the role of responsible investors that are willing to pay a social premium; second, by including activities where the commissioner may attribute a conventional economic value for which he is willing to pay,[2] even though the project outcome in itself does not produce direct economic benefits to him. These considerations may open the way to a broader use of SIBs in new fields in the future. A second SIB limitation is represented by the likelihood of incurring in severe time delays and high costs when evaluating project effects, especially when the best methodologies such as randomised control trials are applied to this purpose. The third caveat is represented by the distortions that may arise from the drive towards the pursuit of an above threshold performance, such as the 'cherry picking' effect, that is, the tendency to select beneficiaries that are most likely to achieve the outcome (OECD, 2016). This limitation can be overcome as far as the SIB is not the only exclusive financial tool providing the service, even though an upward bias in estimating project benefit would remain. Last but not least, SIBs require articulated contracts aimed at reducing conflicts of interest among the various actors involved (see section 3) and, as such, they may have high transaction costs.

Our paper aims at providing an original contribution to the newborn SIB literature by outlining theoretical features of SIBs and discussing two applications. We create a perfect information model where we introduce the private investor's and commissioner's participation constraints by defining the equivalent asset that the financial intermediary purchases when entering into a SIB operation. We find the optimal perfect information SIB contract given by the optimal pair of guarantee fund and share of commissioner profit that maximises the objective function of the latter meeting the private investor participation constraint. We as well discuss the potential conflicts of interest arising among the main SIB actors and move to an imperfect information framework. Then, in the applied section of the paper, we select two existing projects (i.e., jail recidivism and health budget), outline conditions under which the commissioner's and private financier's participation constraints hold, and compute the government multiplier (i.e., the amount saved for any euro invested in the project).

The paper is divided into six sections including introduction and conclusions. In the second section we sketch the perfect information SIB model outlining participation constraints for the government and private financiers. The third section discusses potential conflicts of interest

[1] Three of the most interesting SIB projects around the world have been in this field. The Petersborough project in the UK (Disley et al. 2011 and 2014), the Rikert Island project in the US, and the Juvenile Justice Pay for Success Initiative, Massachussets, US.

[2] An example is provided by job creation where the (implicit) willingness to pay is measured by government expenditure in active labour policies per estimated number of jobs created with those policies.

among SIB actors when removing the assumption of perfect information. In the fourth section we apply the SIB scheme to a project reducing jail recidivism, and in the fifth section we apply it to a health budget project. The sixth section concludes.

2 The perfect information SIB Model

Consider the following social activity (e.g., contrast to jail recidivism) requiring an investment X with expected return $pY+(1-p)F$, where Y is the success outcome, p the probability of success, $1-p$ the probability of failure, and F the failure outcome. The investment in the social activity is risky since $F < X < Y$. In this setting, we interpret the return of the social activity as a reduction of government expenditure aimed to tackle the social problem (e.g., a reduction of jail recidivism implying a reduction of government spending for prisoners). The government is the commissioner of the activity and has to decide whether performing it on its own, or delegating it to a third entity (usually, a non-profit organisation) and using a SIB scheme to finance it. Following SIB rules, we conveniently assume that under the SIB scheme the government raises private capital to cover the investment cost, shares a portion $\pi \in (0,1)$ of the success outcome with the private investor, and creates a guarantee fund $\phi \in (0,1)$ as a share of the investment X, in order to reduce the risk of private investor.

A SIB consists of a pair of (ϕ, π) that satisfies certain government and private investor constraints. Who are the private investors in this scheme? There are three possibilities. First, the government issues a financial asset fractioning the total investment and having exactly the same risk-return characteristics of the aggregate financial investment for each individual investor buying a share of that asset. Hence, each of the n private investors participating to the venture buys exactly $\frac{1}{n}$ of the equivalent asset. Second, private investors are concentrated in one large financial intermediary 'buying' the equivalent asset – this is the case of intermediated SIBs, according to the taxonomy of Goodall (2014) and OECD (2016). Third, the large financial intermediary chooses an 'originate to distribute' model by creating a special purpose vehicle (SPV) issuing a financial asset sold to market investors that yields a fixed interest dividend in case of success and no dividend in case of default. This 'second order equivalent asset' satisfies the participation constraint of individual investors with equality. Under this third scenario, the large financial intermediary has the advantage of covering its costs and getting its margin soon, while distributing the risk on small private investors. The payoff for the FI in this third case can be calculated by subtracting flows with small bondholders and has obviously to be not below the risk-return frontier. The margin of the issuer of the second order equivalent asset has to be considered in this case, thereby leading to higher costs for the operation.

In our context, the government is risk-neutral if it is indifferent between implementing the project by itself and opting for a SIB project. Then, we say that the government is *risk-neutral* if $E_D = E_{SIB}$, and we say that the government is *risk-averse* $E_D < E_{SIB}$.

2.1 Risk-neutrality and the absence of bureaucracy

Government participation constraint under risk-neutrality

To let the private investor participate to the programme, the risk-neutral government wants to expect greater or equal revenues from the SIB than from direct involvement.

The government expected gain in case of direct financing of the social activity is

$$B_{G_D} = p(Y-X) + (1-p)(F-X)$$

while, under the SIB scheme, it writes

$$B_{G_{SIB}} = p(1-\pi)(Y-X) - (1-p)\phi X$$

In essence, the SIB allows the government and private investors to trade part of the risk with a share of profits. The commissioner's participation constraint is therefore met if the expected benefit from choosing the SIB is higher or equal to that of direct financing, that is

$$\begin{aligned} B_{G_{SIB}} &\geq B_G \\ p\pi(Y-X)+(1-p)\big(F-(1-\phi)X\big) &\leq 0. \end{aligned} \tag{1}$$

Equation (1) and its partial derivatives[3] shows that the government incentive to participate lowers with a higher private investors' share π, guarantee fund share ϕ, success outcome Y, and failure outcome F, while increases with a higher investment cost X, coeteris paribus. Moreover, a higher probability of success decreases the government incentive to participate if and only if $\pi(Y-X)-(F-(1-\phi)X)>0$. This assumption is intuitive in our setting, as it requires the government to gain more in case of succes than in case of failure under the SIB scenario. Therefore, we will assume $\pi(Y-X)-(F-(1-\phi)X)>0)$ henceforth.

Thus the 'expected public expenditure multiplier' (i.e., the ratio between the expected value of public expenditure revenues and the expected cost of government participation in the SIB) is given by

$$m_{SIB} := \frac{p(Y-X)(1-\pi)}{(1-p)\phi X}.$$

The private investor participation constraint

From the private investor's point of view, the social activity corresponds to an equivalent asset r with mean

$$\mathbf{E}[r] = p\pi\frac{Y-X}{X} + (1-p)\frac{F-X+X\phi}{X}$$

and standard deviation[4]

$$\sigma^2(r) =: \sigma^2(\pi,\phi) = p(1-p)\frac{1}{X^2}\big(\pi(Y-X)-F+X-X\phi\big)^2. \tag{2}$$

where by abuse of notation we also write $\sigma^2(\pi,\phi)$ to highlight the dependent variables affecting the variance.

The private sector's participation constraint is met if the equivalent asset lies above or along the efficient frontier (EF) (or security market line) represented by

$$\mathbf{E}_{EF}[r] \geq a_0 + a_1\sigma^2(r) \tag{3}$$

that is

$$p\pi(Y-X)+(1-p)(F-X+X\phi) \geq X\Big(a_0 + a_1\frac{p(1-p)}{X^2}\big(\pi(Y-X)-F+X-X\phi\big)^2\Big) \tag{4}$$

where the intercept a_0 and slope a_1 can be estimated using historical nominal rates of return and standard deviations of standard assets such as stocks and bonds. This means that private investors will participate to the venture only if financing the social activity will create an equivalent asset with return and standard deviations that are competitive in financial markets and do not lie below the security market line (4).

[3]The partial derivatives with respect to each variable, that is $\frac{\partial\left(p\pi(Y-X)+(1-p)\left(F-(1-\phi)X\right)\right)}{\partial\pi} > 0$, $\frac{\partial\left(p\pi(Y-X)+(1-p)\left(F-(1-\phi)X\right)\right)}{\partial\phi} > 0$, $\frac{\partial\left(p\pi(Y-X)+(1-p)\left(F-(1-\phi)X\right)\right)}{\partial Y} > 0$, $\frac{\partial\left(p\pi(Y-X)+(1-p)\left(F-(1-\phi)X\right)\right)}{\partial X} < 0$, and $\frac{\partial\left(p\pi(Y-X)+(1-p)\left(F-(1-\phi)X\right)\right)}{\partial F} > 0$

[4]Proof in appendix.

The optimal solution does not exist

Equations (1) and (4) can be jointly satisfied if and only if $\sigma^2(\pi, \phi) = 0$ and $p\pi(Y - X) + (1 - p)(F - X + X\phi) = 0$, which lead to an interesting and unrealistic scenario with zero mean and variance.

2.2 Risk-aversion and absence of bureaucracy

We now assume that the government is risk-averse, that is the government is willing opt for SIB even if this would mean losing part of its expected gain. In our setting, this translates into the following definition: we say that the government is *risk-averse of degree k*, or *k-risk-averse*, if it prefers SIB if and only if

$$k \leq B_{G_{SIB}} \tag{5}$$

for some positive real number k.

The most meaningful and interesting case occurs when $B_{G_D} \geq B_{G_{SIB}} \geq k$, as it allows the government to leave part of its expected gain to the private investors.

Government participation constraint

We have that

$$\begin{aligned} B_{G_D} &= p(Y - X) + (1 - p)(F - X) \\ B_{G_{SIB}} &= p(1 - \pi)(Y - X) - (1 - p)\phi X \end{aligned}$$

and the k-risk-averse government will opt for a SIB if and only if

$$\begin{aligned} B_{G_{SIB}} + k &\geq B_G \\ p\pi(Y - X) + (1 - p)\big(F - (1 - \phi)X\big) &\leq k. \end{aligned} \tag{6}$$

Note that partial derivatives of the government participation constraints are the same as in the risk-neutrality case, and therefore the sensitivity analysis remains unchanged.
Similarly, the expected public expenditure multiplier is unchanged.

The private investor participation constraint

It remains unchanged (see equation (4)).

The optimal solution

Combining (6) and (4), we have two cases.

First, if the government is not risk averse enough (i.e., $k < X(a_0 + a_1\sigma^2(\pi, \phi))$), both constraints will be mutually exclusive and there is no optimal solution.

Second, if the government is risk averse enough (i.e., $k \geq X(a_0 + a_1\sigma^2(\pi, \phi))$, then it would be optimal for it to set its risk aversion coefficient at the minimum, that is $k = X(a_0 + a_1\sigma^2(\pi, \phi))$. In this case the government maximisation problem writes

$$\begin{aligned} &\max_{\pi,\phi} && p(1 - \pi)(Y - X) - (1 - p)\phi X \\ \text{s.t.} \quad &\text{(Gc):} && p\pi(Y - X) + (1 - p)(F - X + X\phi) \leq k \\ &\text{(Pc):} && p\pi(Y - X) + (1 - p)(F - X + X\phi) \geq X(a_0 + a_1\sigma^2(\pi, \phi)) \end{aligned}$$

where (Gc) and (Pc) are respectively the government and the private participation constraint as in (6) and (4).

Now, since $X(a_0 + a_1\sigma^2(\pi,\phi)) = k$ we have

$$\begin{array}{rll} & \max_{\pi,\phi} & p(1-\pi)(Y-X) - (1-p)\phi X \\ \text{s.t.} & \text{(Gc):} & p\pi(Y-X) + (1-p)(F-X+X\phi) \le X(a_0 + a_1\sigma^2(\pi,\phi)) \\ & \text{(Pc):} & p\pi(Y-X) + (1-p)(F-X+X\phi) \ge X(a_0 + a_1\sigma^2(\pi,\phi)) \end{array}$$

that is,

$$\begin{array}{rll} & \max_{\pi,\phi} & p(1-\pi)(Y-X) - (1-p)\phi X \\ \text{s.t.} & \text{(C):} & p\pi(Y-X) + (1-p)(F-X+X\phi) = X(a_0 + a_1\sigma^2(\pi,\phi)) \end{array}$$

and, solving (C), we have

$$\begin{array}{rll} & \max_{\pi,\phi} & p(1-\pi)(Y-X) - (1-p)\phi X \\ \text{s.t.} & \text{(C):} & \pi_{1,2}(\phi) = -\dfrac{Xp + 2a_1p(F-X+X\phi)(1-p) \pm X\sqrt{\frac{p(4a_1(F-X+X\phi)(1-p)+Xp-4Xa_0a_1(1-p))}{X}}}{2(a_1p(X-Y)(1-p))} \end{array}$$

We assume $\pi_1(\phi) > 1$ as it is usually the case in real life context, and we plug π_2 into our maximand, we have

$$\begin{array}{rll} & \max_{\phi} & p(1-\pi_2)(Y-X) - (1-p)\phi X \\ \text{s.t.} & \text{(C):} & \pi_{2}(\phi) = -\dfrac{Xp + 2a_1p(F-X+X\phi)(1-p) - X\sqrt{\frac{p(4a_1(F-X+X\phi)(1-p)+Xp-4Xa_0a_1(1-p))}{X}}}{2(a_1p(X-Y)(1-p))} \end{array}$$

and the optimal SIB writes

$$(\pi^*, \phi^*) = (\frac{a_0X}{Y-X}, \frac{X-F+Xa_0}{X})$$

Note that $\pi^*_{1,2}, \phi^*$ are always non-negative by assumptions on F,X, and Y. However, to be lower or equal to 1, we need the following requirements:

R1 Guarantee fund less than 1 (i.e., $\phi^* \le 1$) requires $a_0 \le \frac{F}{X}$;

R2 Gain share less than 1 (i.e., $\pi^*_2 \le 1$) requires $a_0 \le \frac{Y-X}{X}$;

Comparative statics

A comparative static analysis shows that

- $\frac{\partial \pi_2*}{\partial X} = \frac{a_0(Y-2X)}{(Y-X)^2} > 0 \iff Y > 2X$
- $\frac{\partial \pi^*_2}{\partial Y} = -\frac{a_0X}{(Y-X)^2} < 0$
- $\frac{\partial \pi^*_2}{\partial a_0} = \frac{X}{Y-X} > 0$
- $\frac{\partial \phi^*}{\partial F} = -\frac{1}{X} < 0$
- $\frac{\partial \phi^*}{\partial X} = \frac{F}{X^2} > 0$
- $\frac{\partial \phi^*}{\partial a_0} = 1 > 0$
- The guarantee fund does not depend on Y
- The solutions do not depend on a_1 and p

Government multiplicator

The 'expected public expenditure multiplier' (i.e., the ratio between the expected value of public expenditure savings and the expected cost of government participation in the SIB) writes

$$\frac{B_{G_{SIB}}}{B_G - B_{G_{SIB}}} = \frac{p(1-\pi)(Y-X) - (1-p)\phi X}{p\pi(Y-X) + (1-p)(F-X+X\phi)} \tag{7}$$

3 Potential conflicts of interest under an asymmetric information scenario

The above described comparative statics discloses several conflicts of interest that may arise if we relax the assumption of perfect information. Asymmetric information may arise under different respects, such as risk-return characteristics of the activity and quality and effort of the delegated organisation performing the social service.

On the first point, the organisation performing the service would be interested in increasing project costs, as project costs are indeed revenues for the organisation (Table 1). An independent audit on project costs may be required by the government and private investors.

Table 1: Potential conflicts of interest under the SIB scheme.

	Government	**Private investors**	**NGO**	**Solution**
Project cost	Interest to reduce project costs to pay less in terms of guarantee fund	Interest to reduce project costs in order to increase project profits	Interest to inflate project costs (avoiding that it passes down the trigger point and the SIB is not done) as they are proportional to their wages	Cost sharing for NGO
Project expected revenues	Interest to inflate revenues to tell private investors that they have a high return project so that they accept higher risk ... offset by the problem of higher guarantee fund	Interest to reduce revenues	Interest to inflate revenues up to the trigger point that makes the SIB feasible	
Project expected risk	Interest to reduce risk in order to give lower profit share and lower guarantee fund coverage	Interest to increase risk in order to get higher return (i.e., profit share) or higher guarantee fund coverage	Interest to reduce risk up to the trigger point that makes the SIB feasible	Evaluation of the project risk from independent third parties
Choice of NGO	Interest for a political friendly less efficient NGO (votes more important than public debt)			NGO chosen by private investor

On the second point, government officials may be politically biased, that is, they may be interested in selecting the organisation ensuring the highest political benefits, and not necessarily the best performer. This may happens because government officials do not directly incur in costs in case the project fails.[5] In this case, it is advisable that the selection of the organisation be in charge of private investors, who directly benefit from the success of the venture.

Similarly, private investors are interested to overstate project risks in order to negotiate a higher share of profits from the commissioner. In fact, they may declare a higher level of risk such that the profit share π making their participation constraint hold is higher than the actual profit share associated with the effective level of risk. On the contrary, the government and the organisation performing the service have the opposite interest – they aim at to showing that the project is feasible and private financing is profitable (Wang et al. 2013). As for the previous case, an audit of a third independent party of risks and returns of the project can overcome these problems.

Two additional conflicts of interest that can typically occur concern hidden action of the service provider – when her/his effort cannot be monitored – and hidden information on project output. This does not apply on the direct output of the project (e.g., in section 4 we provide an example of jail recidivism where we know exactly how many prisoners return to jails or another example of "health budgets" where we know whether patients are re-hospitalised or not). It however applies on the counterfactual the project output is compared with, in order to assess the success or the failure of the project. In other words, the benchmark value of the output variable under the scenario without the treatment can be arguable. The first problem may be overcome with some form of variable (i.e., performance based) payment to service providers; the second problem may be overcome with an ex ante agreement between commissioners and the intermediary on the counterfactual (e.g., the regional average return to jail rate in the case of the jail recidivism example).

4 The SIB in action: The case of jail recidivism

The Made in Carcere (MiC) project trains inmate women in the craftsmanship sector with the goal of reintegrating them in the job market. The project has been tested for the first time in 2007 in Puglia, Italy for 10 years with a group of 123 women. Trainees have produced handcrafted clothes branded as MiC, products have been sold, and market discipline has helped convicted women to develop job discipline and to improve their skills and competences, hence their professional qualities and productivity. The project has reduced jail recidivism in Lecce (South of Italy) from 70 to 5 percent in 10 years.

A SIB scheme can replicate this project on a larger scale with significant benefits for the government budget. Government gains on the projects are represented by the forgone costs amounting to 58,000 euros for one year in jail.[6]

[5] In some specific legislations, civic servants can be prosecuted and found directly responsible with their own wealth for damages to public money. Even in that case the expected costs of their damage action may be low in case of poor efficiency of civil justice.

[6] These costs represent the ratio between total government expenditure for the prison system industry and the number of prisoners. Hence they combine variable and fixed costs. As such we are aware that, while variable costs may be related more directly to the number of prisoners, fixed costs may be saved if the reduction is permanent and of a scale allowing to eliminate the need of one prison infrastructure. Given the large number of prisoners involved in our SIB scheme, it is reasonable to assume that the number is high enough to imply savings also on fixed costs and jail infrastructure. The other implicit assumption is that of zero queues, a reasonable assumption for jails. The problem is generally the opposite in most countries. The example in the country of the MiC experiment (Italy) is a 113% ratio between effective prisoners and the maximum admissible number according to EU rules in 2017 – see http://www.repubblica.it/solidarieta/diritti-

In our best case scenario we replicate the project by considering yearly fixed costs of 200,000 euros. These costs include payment of resources employed in women training based on real costs of the first MiC project, plus a bonus calculated as 10 percent of revenues for prison guards (Table 2). We calculate values for the overall period equivalent asset using a discount rate of 5 percent. As a best case scenario, we consider a share of project profits for private investors equal to 20 percent (i.e., the government retains 80 percent) and a guarantee fund of the state covering 20 percent of the investment cost. We assume that the project is successful (i.e., it replicates the success story of MiC with a reduction of recidivism from 70 to 5 percent) with probability 0.8 (good state), and fails with probability 0.2 (bad state) (ie. because it may be difficult to replicate on a larger scale the outcome of the experimental project). We conveniently assume that the bad state is represented by a 20 percent loss of capital invested implying a reduction of recidivism of only 8 percent, 3 percent more than the 5 percent reduction of the no treatment scenario. The loss under the bad state is covered by the guarantee fund, that covers exactly 20 percent of the investment cost, so that private investors have a 0 percent return on their investment but do not lose their capital. We as well assume that average years in jail post recidivism are three[7] and that the effect of recidivism reduction produced by the project is uniformly distributed over 10 years (i.e., any year the positive effect of recidivism reduction is produced on 10 percent of the treated). This implies also an assumption of uniform distribution of the remaining jail years for participants to the project. The 3-year recidivism assumption produces three different revenues for the first year asset, second year asset, and third year asset, since government savings in the third year of the project are three times higher than in the first.

In this best case scenario, the equivalent asset is above the risk-return frontier and the government participation constraint is met. The project is therefore viable. More specifically, the 12-year compounded return is 310 percent corresponding to a yearly return of 12.5 percent. The overall period equivalent asset (80 percent probability of 12.5 return and 20 percent probability of 0 percent return) has standard deviation of 5.27 and is well above the risk-return frontier (the return on the frontier for that standard deviation is 2.12 percent). The government guarantee fund amounts to 516,900 euros and mobilises a private investment of 3.252 millions. Total (non-discounted) government gains over 12 years are 7,629 millions. Considering the 20 percent default risk the guarantee fund could be used with multiplier 5 but, for reasons of prudence, the multiplier could be limited to 2. The public expenditure multiplier is 14.75 (twice of it if we use the guarantee fund to finance 2 MiC projects).

Note that, if we decompose the equivalent asset into three separate assets (i.e., first, second and third year asset) and we consider that revenues are different in each year, we find that every asset is far above the efficient market line.

The SIB is viable for private investors also in case the third scheme is used, that is, the holder of the first order equivalent asset issues a bond (second order equivalent asset) to distribute part of the risk on small bondholders. We assume costs of issuance at 1 percent of the total amount issued. The issuer offers a bond paying a 1.5 percent yield in case of success and no yield in case of failure (20 percent probability) the first year. The success yield raises to 2 percent in

umani/2017/07/31/news/carceri_in_italia_crescono_pericolosamente_sovraffollamento_e_suicidi-172043754/. For this reason Italy is subject to a fine from the EU. Savings on EU fine costs are not added to the picture that therefore may underestimate actual benefits from the MIC project.

[7]Three years in jail post recidivism are a compromise incorporating considerations about the average expected number of post recidivism years in jail of the treated without treatment, duration, and drop-off of the treatment. In our sensitivity analysis we check whether the SIB remains feasible when we relax this assumption and consider 2 or 1 year of post recidivism jail. Note however that recidivism implies usually more severe penalties for crime reiteration vis-á-vis the first prosecuted crime. Hence our base assumption may even be underestimated when we take this into account.

the second year and 3 percent in the third year. The net return of the issuer is the difference between the first order equivalent asset yield and the second order equivalent asset yield paid to bondholders minus the transaction costs. Both (first and second order) equivalent assets are well above the security market line in all of the three years. Note that, under this third scheme, the standard deviation of the equivalent asset is lower for both the issuer and the bondholders, consistently with the originate-to-distribute mechanism created.

As always in impact studies, it is fundamental to evaluate whether project benefits are overestimated for not taking into account deadweight, crowding-out, attribution, and drop-off. Deadweight is represented in our case by the complement of the recidivism rate without intervention. Therefore, it is already considered since we calculate project gain as the difference between the recidivism rate with and without the project. As well, there is no crowding out because we assume there are no other projects alternative to the standard public jail path in absence of the MiC project. The result of the project can attributed 100 percent to the treatment. Drop-off is already implicit in our assumption of average recidivism length (e.g., if average length is three years the treatment has full effects for three years and 100 percent drop-off after them).

After showing the feasibility of our best case scenario we apply to it the optimal solutions developed in the theoretical section. Under this best case scenario the optimal government guarantee fund is 39.6 percent, while the optimal share of profits to the private investor is 44.9 for the first year asset, 18.27 for the second year asset and 12.67 for the third year asset. [qui dobbiamo parlare dell'ottimo che ci viene fuori con massimizzazione !! e di che dati produce sugli altri parametri quanto utile dello stato in questo caso ?]

4.1 Sensitivity analysis

Our best case scenario may be considered excessively optimistic when looking at critical parameters. We perform sensitivity analysis on the four of them that we consider most relevant: i) fixed costs (raised from 200,000 to 400,000); ii) probability of success (reduced down from the 80 percent base assumption); iii) average years in jail post recidivism (reduced from 3 to 2 or 1 year only); iv) loss in bad state.

If we increase fixed costs, we find that government and private participation constraints are still met up to a maximum of 400,000 euros of fixed costs per year (twice as much as in our best case scenario). In this case the share of profits to private investors may fall down to 13 percent and still meet the government participation constraints, given that now the benefit of a SIB scheme for the government grows due to the higher foregone costs in case of project failure (Table 3, columns 2 and 3).

We also find that the SIB scheme remains feasible when departing from the best case scenario with the reduction of average recidivism years to 2, and even when we combine this change with an increase in fixed costs up to 300,000 euros (Table 4, column 2). If fixed costs grow to 400,000 euros, combined with the assumption of 2 average years of post recidivism jail, the private investors' participation constraint requires a share of profits of at least 20 percent to be met (Table 4, column 3).

The SIB scheme remains viable under the assumption of only one average year of post recidivism jail and 300,000 euros of fixed investment costs (Table 4, column 5). The share of profits to private investors may rise up to 50 percent and still meet the government participation constraint.

The extreme bound of relaxation of our hypothesis on fixed costs and post recidivism years

Table 2: Best case scenario for a SIB in the Made in Carcere project

	Made in Carcere
Intertemporal discount rate	5 percent
Total operating costs per year	200,000
Bonus for prison guards (% of revenues)	10 percent
Average years of recidivism	3
Project length	10 years
Distribution of the effect	Linear (10 percent per year)
Prob. of good state	80 percent
Prob. of bad state	20 percent
Reduction of recidivism in good state	From 5 (average in the no treatment scenario) to 70 percent
Reduction of recidivism in bad state	8 percent
Government guarantee fund	20 percent
Loss of private investors' capital under bad state	-20 percent (fully covered by guarantee fund)
Share of government revenues given to private investors	5 percent
Yearly return of the first year asset (percent)	3.53
Standard deviation of the first year asset	1.86
Yearly Return of equivalent (first year) asset on the risk-return frontier (percent)	1.03
Yearly Return of the second year asset (percent)	8.67
Standard deviation of the second year asset	4.57
Yearly return of equivalent (second year) asset on the risk-return frontier (percent)	1.90
Yearly Return of the third year asset (percent)	12.41
Standard deviation of the third year asset	6.54
Yearly Return of equivalent (third year) asset on the risk-return frontier (percent)	2.53
Government multiplier	14.75
Government multiplier (with guarantee fund multiplier of 2)	29.5

in jail is doubling fixed costs (up to 400,000 euros) and reducing to one average post recidivism years in jail. Under this scenario private investors would participate with a 20 percent profit share and the government with a guarantee fund of 20 percent of the investment cost. The government participation constraint is met but the multiplier is below one.

While the viability of the SIB scheme is quite resistant to more pessimistic hypotheses on fixed costs and average post-recidivism years in jail, it results to be much more sensitive to more pessimistic hypotheses on the risk of default (above 20 percent) and on the downside risk (up from the 40 percent of the best case scenario, half of which covered by the guarantee fund). More specifically, the SIB scheme is still viable up to a 40 percent loss in case of default everything else being equal, or up to a 40 percent loss, but only if the guarantee fund of the government raises to 30 percent. Note that the government guarantee fund cannot raise more compatibly with the government participation constraint.

As well, keeping everything else constant, the probability of default cannot overcome the 30 percent threshold in order to make the project still viable for private investors that are bearing most of its risk.

4.2 Further discussion

In our analysis, we strictly limit project benefits to the foregone government costs of recidivism. However, the MiC project has other positive effects on the lives of the treated. First, women that do not go back to jail may work and therefore their wage is a monetary proxy of such benefit. As well, if they can reconcile their work with family life (and if they have children), they can care about them avoiding their families to pay market services for them. In order to avoid double counting, we can limit these benefits to only one of the two effects (we can assume that the treated either take care of their children or find a new job). We did not count these monetary benefits in project profits as they are not government gains (in terms of reduced government expenditure for prisoners in jail). However, it is highly likely that the government gives value to these monetary benefits and is willing to pay for them. Any time a public administration invests in active employment policies it is in fact implicitly 'spending' a given amount of money per job created (the ratio between total active employment policy expenditure and the number of jobs created). Hence, it is possible that the government agrees to pay a given amount to SIB counterparts for any job created.

5 The SIB in action: The Health Budget project

The Health Budget project is a three-year personalised plan activated in Campania (Italy) in the last decade by a team of doctors and psychologists on patients with mental diseases. The project is an alternative to hospitalisation in a psychiatric public structure. The team studies personalised plans for each patient, proposing innovative solutions such as work reintegration and social farming. In particular, nowadays social farming offers widespread opportunities and also multinationals have joined the project, including Leroy Merlin with its CSR program in Italy. The project has been tested the first time on a target of 60 patients. In this first trial, the work of three professionals (psychologists and doctors) costed 82 euros per patient per day, versus a daily cost of 300 euros under the hospitalisation alternative. The project was fully successful, since none of the patients have been re-hospitalised in the following years.

In this section, we create a unit SIB simulation which replicates the project on the same number of patients. The total cost of the project for three years (calibrated on the observed cost of 82 euros per day as described above) is 5,387 million euros. We prudentially and drastically cut the success rate of the project in the good state by assuming that it works only for 50 percent

Table 3: Sensitivity analysis for a SIB in the Made in Carcere project – changes in fixed costs

Intertemporal discount rate	5 percent	5 percent	5 percent
Total operating costs per year	200,000	300,000	400,000
Bonus for jail (% of revenues)	10	10%	10%
Avg years of recidivism	3	3	3
Project length	10ys	10ys	10ys
Distribution of the effect	Linear (10% per year)	Linear (10% per year)	Linear (10% per year)
Prob. of good state	80%	80%	80%
Prob. of bad state	20%	20%	20%
Reduction of recidivism in good state From average (70%) to 5%	From average (70%) to 5%	From average (70%) to 5%	
Reduction of recidivism in bad state	X	X	X
Govmt guarantee fund	20%	20%	20%
Loss of private investors capital under bad state	-20% (fully covered by guarantee fund)	-20% (fully covered by guarantee fund)	-20% (fully covered by guarantee fund)
Share of govmt revenues given to private investors	5%	5%	5%
Yearly Return of the first year asset (percent)	3.53%	1.35%	0.15%
Std dev. First year asset	1.86	0.71	0.08
Yearly Return of eq. asset on the risk-return frontier (percent)	1.03	0.66	0.46
Yearly Return of the second year asset (percent)	8.67	5.44	3.53
Std dev. Second year asset	4.57	2.87	1.86
Yearly Return of eq. asset on the risk-return frontier (percent)	1.90	1.35	1.03
Yearly Return of the third year asset (percent)	12.41	8.67	6.32
Std dev. third year asset	6.54	4.57	3.33
Yearly Return of eq. asset on the risk-return frontier (percent)	2.53	1.89	1.50
Government multiplier	14.75	10.13	7.28
Government multiplier (with guarantee fund multiplier of 2)	29.5	20.26	14.56

Table 4: Sensitivity analysis for a SIB in the Made in Carcere project – changes in recidivism years

Intertemporal discount rate	5%	5%	5%	5%	5%	5%
Total operating costs per year	200,000	300,000	400,000	200,000	300,000	400,000
Bonus for jail (% of revenues)	10%	10%	10%	10%	10%	10%
Avg years of recidivism	2 years	2 years	2 years	1 year	1 year	1 year
Project length	10	10	10	10	10	10
Distribution of the effect	Linear	Linear	Linear	Linear	Linear	Linear
Prob. of good state	80%	80%	80%	80%	80%	80%
Prob. of bad state	20%	20%	20%	20%	20%	20%
Reduction of recidivism in good state	70	70	70	70	70	70
Reduction of recidivism in bad state	5	5	5	5	5	5
Govmt guarantee fund	20	20	20	20	20	20
Loss of private investors capital under bad state	20	20	20	20	20	20
Share of government revenues given to private investors	5	5	20	20	50	20
Yearly Return of the first year asset (percent)	3.53	1.36	0.62	14.11	13.55	0.621
Std dev. First year asset	1.86	0.71	0.33	7.44	7.14	0.327
Yearly Return of eq. asset on the risk-return frontier	1.02	0.66	0.54	2.81	2.52	0.538
Yearly Return of the second year asset (percent)	8.67	5.44	14.11			
Std dev. Second year asset	4.57	2.87	7.44			
Yearly Return of eq. asset on the risk-return frontier (percent)	1.90	1.35	2.81			
Gvmt multiplier	10.21	6.35	3.43	3.52	0.15	
Gvmt multiplier (with guarantee fund multiplier of 2)	20.42	12.7	6.86	7.04	0.30	

of the targeted patients [8]. We as well calculate that the project is successful with probability 80 percent, while it fails with probability 20 percent with financiers incurring in a 20 percent loss under this bad state of affairs (half of which is covered by the guarantee fund). Note that gains are immediately available as the project takes patients away from the structure since its start. As well, consider that gains for the government should last for all the rest of the patient's life. We however very conservatively assume an extreme hypothesis of lack of government benefits at the end of the three years. This implies either an implausible zero life expectancy after the end of the treatment or a more realistic hypothesis of partial rationing on the number of beds in hospital psychiatric structure (hence the benefit of bringing the treated patients away is reduced since another patient in the list enters the structure).

Given these parameters and our model the solution of the government maximisation problem under the assumption of its minimal risk aversion level is an optimal share of profits to private investors of 35 percent and a guarantee fund equal to 93 percent.

Under this best case scenario the government total discounted gains (using a 5 percent discount rate) for the three years are 1.68 millions and the total three year return of the project is 32.9 percent. The 'equivalent asset' purchased by private investors (including the 20 percent probability of -20 percent returns, where part of the loss is covered by the government guarantee fund) has therefore a (yearly)[9] standard deviation of 13.98 and a (yearly) mean return of 5.22 percent, above the 4.69 percent return on the risk-return frontier point corresponding to that standard deviation. Return and standard deviation of the equivalent asset are calculated only on the three years in which the project incurs in costs and do not consider benefits for the following years in which there are no costs and the project continues to produce benefits.

The government participation constraint (under the hypothesis of minimal risk aversion) is as well met since government profits are higher under the SIB than under direct investment.

The total multiplier of public expenditure (i.e., the ratio between the government expenditure represented by the guarantee fund and the ex post government gains net out of dividends to private investors) is 6.63 (the government guarantee fund is of 1,077 mln and generates profits for the government share of around 7,148 mln). The multiplier can be doubled if we consider that the default probability of 20 percent allows for a maximum multiplier for the guarantee fund of 5 that can be prudentially reduced to 2 (that is the same guarantee fund can be used for 2 projects).

5.1 Sensitivity analysis on the health budget project

Our sensitivity analysis explores the conditions that the health budget project requires to be viable when we reduce the daily cost per patient under the no treatment care – and so government gains per patient in the health budget project. This analysis is particularly relevant since it allows us to check whether the SIB scheme is viable also in those regions where the cost of patients held in public structures is lower (and not up to 300 euros per day including drugs and medical treatment as in the two Italian regions where the Health Budget experiment was first realised, i.e., Campania and Lazio). Our findings in Table **??**, column 3, show that – cœteris paribus – daily costs can fall only from 300 to 280 euros, while share of profits to private investors to meet their participation constraint contemporarily raising from 40 to 60 percent, thereby making them more binding.

[8] Again this may be due to the difficulty of replicating the project on large scale that implies higher supply of external opportunities for the patients.

[9] Given that the asset has exactly the same characteristics in each investment year, for the sake of simplicity, we calculate return and standard deviation at year level in order to check whether the private investor participation constraint is met.

Table 5: Best case scenario for a SIB in the Health Budget project

	Health budget
Intertemporal discount rate	5 percent
Total operating costs	82 per day vs 300 per day
Success rate (% of successful patients)	50 percent
Average years of treatment	3
Years of project effects after treatment	4
Good state	80 percent
Bad state	20 percent
Loss in bad state	20 percent
Government guarantee fund	20 percent
Share of government revenues given to private investors	40 percent
Yearly return of the equivalent asset (percent)	5.2
Std dev. equivalent asset	13.98
Yearly return of eq. asset on the risk-return frontier (percent)	4.69
Government multiplier	5.43
Government multiplier (with guarantee fund multiplier of 112)	10.86

Table 6: Sensitivity analysis for a SIB in the Health Budget project

Intertemporal discount rate	5%	5%	5%	5%	5%
Total operating costs per day	82 vs 300	82 vs 280	250	200	300
Success rate under good state (% of successful patients)	50%	80%	80%	70%	
Avg. years of treatment	3	3	3	3	
Yrs of project effects after treatment	4	4	4	4	
Good state	80%	80%	80%	80 %	80 %
Bad state	20%	20%	20%	20%	20%
Loss in bad state	40%	40 %	40 %	40 %	50 %
Gov'nt. guarantee fund	20%	20%	20 %	20 %	30 %
Share of gov'nt revenues given to private investors	40%	60%	30%	80%	20%
Yearly return of the equivalent asset (percent)	6.54	5.91	11.34	5.78	7.78
Std. dev. equivalent asset	13.98	13.68	16.52	13.53	19.90
Yearly return of eq. asset on the risk-return frontier (percent)	4.90%	4.81	5.71	4.76	6.80
Government multiplier	6.63	5.66	9.11	5.21	6.35
Government multiplier (with guarantee fund multiplier of 2)	13.26	11.32	18.22	10.42	12.7

What we find is that, with a daily cost per patient below 280 euros, the project is feasible only if it departs from our best case scenario hypothesis of success under the good state by raising it from 50 to 80 percent. In this case, the project is still feasible when the daily cost of the standard structure falls up to 200 per patient (Table x, columns 4-5). Note that the probabilities that our scenario assumes make our analysis very conservative, given that in the first experimental trial in the project has been successful for the 100 percent (all the 60 patients have been successfully treatment and have not gone back to the structure).

The other very sensitive parameter is, again, the downside loss under the worst case scenario. If we raise the loss under the bad state (i.e., at 60%) there is no way to make the project feasible for both government and private investors participation constraints. In fact, we may raise the private share of profits to make the project viable for private investors but, in that case, the project is no more feasible for the government. The only way to make the project feasible is to raise the government guarantee fund to 30 percent, with a probability of success of the good state at 70% and a share of profits to private investors of 20%. This last sensitivity check confirms that project risk is the most sensitive variable that may prevent SIB to be viable.

6 Conclusions: what we have learnt

SIBs are innovative promising financial schemes involving several actors. Under the SIB scheme the most efficient and reputable organisation in the provision of a given social service that reduces government expenditure is hired by the government, and private investors participate to the venture by financing it with their funds. Investment risk is therefore rolled over them by the government that only partially covers the risk with a guarantee fund. In case of success, government gains in terms of reduced public expenditure are shared with private investors.

In this paper we show that the very (perfect information) SIB problem consists in the government and private investors contracting profit shares and the share of investment covered by a guarantee fund reimbursing private investor losses. The scheme is viable and convenient if both the government and private investors participation constraints hold. More specifically, the SIB scheme is convenient for private investors when their participation is equivalent to purchasing an equivalent asset not below the security market line; it is convenient for the government when the it ensures higher gains upon the alternative of direct financing. This can happens through the following mechanism: SIB leverages private capital transferring on it part of the risk, as it mobilises a limited share of government resources up to the amount of the guarantee fund. An important advantage of the scheme consists of generating a high public expenditure multiplier that improves its budget, as the share between public expenditure "revenues" (savings on government expenditure for the social service) and public resources immobilised in the guarantee fund is high. Our paper also shows that risk moderation (with the guarantee fund) plays a crucial role to satisfy the private investor participation constraint, while limited participation of private investors to project profits plays a crucial role to satisfy the government participation constraint.

The SIB is a complex and articulated infrastructure involving actors with different objective functions. Therefore, it requires well-designed governance and rules when we depart from the perfect information framework. In particular, we argue as advisable that the private investor takes part to the selection of the organisation performing the social task, in order to avoid political bias when the selection is performed by the government. We also consider that an audit of independent third parties is essential to ascertain project revenues, costs, return, and risk in order to avoid distortion in their evaluation by one of the involved parties for their own interest. We as well discuss other two imperfect information problems arising in the scheme such as hidden information on the final project outcome and hidden action of the organisation performing the social service.

In the second part of the paper, we apply the SIB structure by simulating the replication on larger scale of the figures of two projects realised in the past that have particularly promising features (i.e., a project aimed to prevent jail recidivism and a health budget project). We find that, given current standard cost parameters for service provision by the government, SIB schemes for the two projects are viable in that they ensure above security market line risk-adjusted profits for private investors, while meeting at the same time the government participation constraint. We as well show that the participation constraint of private investors holds not only with their direct financing for a single financial intermediary on the private investors' side, but also under an originate-to-distribute model where a Special Purpose Vehicle redistributes the risk on a large number of small private investors.

Results of our paper provide a theoretical and empirical framework to develop and apply SIBs schemes to different types of social services and can stimulate further contributions in this novel field of the literature.

References

[1] Disley, E., J. Rubin, E. Scraggs, N. Burrowes, and D. Culley (2011). *Lessons learned from the planning and early implementation of the Social Impact Bond at HMP Peterborough.* London: Ministry of Justice.

[2] Disley, E. and J. Rubin (2014). *Phase 2 report from the payment by results Social Impact Bond pilot at HMP Peterborough.* London: Ministry of Justice. Retrieve from https://www.gov.uk/government/uploads/system/uploads/attachment_data/file/325738/ peterborough -phase-2-pilot-report.pdf.

[3] Gustafsson-Wright, E., S. Gardiner, and V. Putcha (2015). Potential and limitations of impact bonds: Lessons from the first five years of experience worldwide, Global Economy and Development Program, Brookings Institution. Retrieved from http://www.brookings.edu/∼/media/Research/Files/Reports/2015/07/social-impact-bonds-potentiallimitations/Impact-Bondsweb.pdf?la=en

[4] OECD (2016). *Understanding Social Impact Bonds.* OECD Working Papers, Paris, France.

A Appendix

Proof of equation (2).

$$
\begin{aligned}
\sigma_a(r) &= E_a[r^2] - E[r]^2 = \\
&= p\pi^2(\frac{Y-X}{X})^2 + (1-p)(\frac{F-(1-\phi)X}{X})^2 \\
&\quad -p^2\pi^2(\frac{Y-X}{X})^2 - (1-p)^2(\frac{F-(1-\phi)X}{X})^2 - 2p(1-p)\frac{Y-X}{X}\frac{F-(1-\phi)X}{X} = \\
&= p(1-p)\pi^2(\frac{Y-X}{X})^2 + p(1-p)(\frac{F-(1-\phi)X}{X})^2 - 2p(1-p)\frac{Y-X}{X}\frac{F-(1-\phi)X}{X} = \\
&= p(1-p)\Big(\pi^2(\frac{Y-X}{X})^2 + (\frac{F-(1-\phi)X}{X})^2 - 2\frac{Y-X}{X}\frac{F-(1-\phi)X}{X}\Big) = \\
&= p(1-p)\Big(\pi\frac{Y-X}{X} - \frac{F-(1-\phi)X}{X}\Big)^2
\end{aligned}
$$

□

A.1 (C1) is a parabola

We show that constraint (C1) is a parabola in the $\pi\phi$-plane.

Proof. In terms of π and ϕ, (C1) writes

$$
\begin{aligned}
&a_1pq\frac{1}{X^2}(Y-X)^2\pi^2 + a_1pq\phi^2 - 2a_1pq\frac{1}{X}(Y-X)\pi\phi + \\
&-p\frac{Y-X}{X}(1+2a_1q(\frac{F}{X}-1))\pi - q(1-2a_1p(\frac{F}{X}-1))\phi + q(\frac{F}{X}-1) + a_0 + a_1pq(\frac{F}{X}-1)^2 = 0
\end{aligned}
$$

According to the classification of conics, we name coefficients as elements of the matrix $N = \begin{pmatrix} m_{11} & m_{12} & n_1 \\ m_{21} & m_{22} & n_2 \\ n_1 & n_2 & k \end{pmatrix}$, that is

$$
\begin{aligned}
&\underbrace{a_1pq\frac{1}{X^2}(Y-X)^2}_{m_{11}}\pi^2 + \underbrace{a_1pq}_{m_{22}}\phi^2 - 2\underbrace{a_1pq\frac{1}{X}(Y-X)}_{m_{12}}\pi\phi - 2\underbrace{\frac{1}{2}p\frac{Y-X}{X}(1+2a_1q(\frac{F}{X}-1))}_{n_1}\pi + \\
&-2\underbrace{\frac{1}{2}q(1-2a_1p(\frac{F}{X}-1))}_{n_2}\phi + \underbrace{q(\frac{F}{X}-1) + a_0 + a_1pq(\frac{F}{X}-1)^2}_{k} = 0
\end{aligned}
$$

We show that the determinant of the submatrix $M = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}$ is equal to zero.

$$
\det M = \frac{1}{X^2}a_1pq(Y-X)^2 \cdot a_1pq - \frac{1}{X}a_1pq(Y-X) \cdot \frac{1}{X}a_1pq(Y-X) = 0
$$

Moreover, $\det N = \cdots = -\frac{1}{4}\frac{1}{X^4}a_1pq(Y-X)^2$, and $\det N \neq 0 \iff a_1, p, q, X \neq 0$ and $Y \neq X$, **which holds by assumption**. □

A.2 Solving the maximisation problem

The maximisation problem can be written as

$$
\begin{array}{lll}
 & \max_{\pi,\phi} & -p(Y-X)\pi-(1-p)X\phi+p(Y-X) \\
\text{s.t.} & \text{(c1):} & a_1p(1-p)\dfrac{(Y-X)^2}{X^2}\pi^2+a_1p(1-p)\phi^2-2a_1p(1-p)\dfrac{Y-X}{X}\pi\phi+ \\
 & & -p\dfrac{Y-X}{X}(1+2a_1(1-p)\dfrac{F-X}{X})\pi-(1-p)(1-2a_1p\dfrac{F-X}{X})\phi+ \\
 & & +(1-p)\dfrac{F-X}{X}+a_0+a_1p(1-p)\dfrac{(F-X)^2}{X^2}=0 \\
 & \text{(c2):} & -\pi\leq 0 \\
 & \text{(c3):} & \pi\leq 1 \\
 & \text{(c4):} & -\phi\leq 0 \\
 & \text{(c5):} & \phi\leq 1
\end{array}
$$

and the Lagrangian writes

$$
\begin{aligned}
L(\pi,\phi,\lambda,\mu_1,\mu_2) \;=\;& -p(Y-X)\pi-(1-p)X\phi+p(Y-X)+ \\
& +\lambda\Big(a_1p(1-p)\frac{(Y-X)^2}{X^2}\pi^2+a_1p(1-p)\phi^2-2a_1p(1-p)\frac{Y-X}{X}\pi\phi+ \\
& -p\frac{Y-X}{X}(1+2a_1(1-p)\frac{F-X}{X})\pi-(1-p)(1-2a_1p\frac{F-X}{X})\phi+ \\
& +(1-p)\frac{F-X}{X}+a_0+a_1p(1-p)\frac{(F-X)^2}{X^2}\Big)+ \\
& +\mu_1(-\pi)+\mu_2(\pi-1)+\mu_3(-\phi)+\mu_4(\phi-1)
\end{aligned}
$$

The Karush-Kuhn-Tucker conditions write

$$
\begin{cases}
\frac{\partial L}{\partial\pi}: -p(Y-X)+\lambda\Big[2a_1p(1-p)\frac{(Y-X)^2}{X^2}\pi-2a_1p(1-p)\frac{Y-X}{X}\phi-p\frac{Y-X}{X}(1+2a_1(1-p)\frac{F-X}{X})\Big]-\mu_1+\mu_2=0 \\
\frac{\partial L}{\partial\phi}: -(1-p)X+\lambda\Big[2a_1p(1-p)\phi-2a_1p(1-p)\frac{Y-X}{X}\pi-(1-p)(1-2a_1p\frac{F-X}{X})\Big]-\mu_3+\mu_4=0 \\
\frac{\partial L}{\partial\lambda}: a_1p(1-p)\frac{(Y-X)^2}{X^2}\pi^2+a_1p(1-p)\phi^2-2a_1p(1-p)\frac{Y-X}{X}\pi\phi-p\frac{Y-X}{X}(1+2a_1(1-p)\frac{F-X}{X})\pi+ \\
\quad -(1-p)(1-2a_1p\frac{F-X}{X})\phi+(1-p)\frac{F-X}{X}+a_0+a_1p(1-p)\frac{(F-X)^2}{X^2}=0 \\
\mu_1(-\pi)=0 \\
\mu_1\geq 0 \\
\mu_2(\pi-1)=0 \\
\mu_2\geq 0 \\
\mu_3(-\phi)=0 \\
\mu_3\geq 0 \\
\mu_4(\phi-1)=0 \\
\mu_4\geq 0
\end{cases}
$$

If we (reasonably) exclude corner solutions (Possiamo assumerlo dall'inizio???), that is $0<\pi<1$ and $0<\phi<1$, then $\mu_1,\mu_2,\mu_3,\mu_4=0$ and we can write

$$\begin{cases} (1): & -p(Y-X)+\lambda\Big[2a_1p(1-p)\frac{(Y-X)^2}{X^2}\pi-2a_1p(1-p)\frac{Y-X}{X}\phi-p\frac{Y-X}{X}(1+2a_1(1-p)\frac{F-X}{X})\Big]=0\\ (2): & -(1-p)X+\lambda\Big[2a_1p(1-p)\phi-2a_1p(1-p)\frac{Y-X}{X}\pi-(1-p)(1-2a_1p\frac{F-X}{X})\Big]=0\\ (3): & a_1p(1-p)\frac{(Y-X)^2}{X^2}\pi^2+a_1p(1-p)\phi^2-2a_1p(1-p)\frac{Y-X}{X}\pi\phi-p\frac{Y-X}{X}(1+2a_1(1-p)\frac{F-X}{X})\pi+\\ & \quad -(1-p)(1-2a_1p\frac{F-X}{X})\phi+(1-p)\frac{F-X}{X}+a_0+a_1p(1-p)\frac{(F-X)^2}{X^2}=0 \end{cases}$$

We multiply (1) by $\frac{X}{Y-X}$ (we can as $Y>X$)

$$\begin{cases} (1'): & -pX+\lambda\Big[2a_1p(1-p)\frac{Y-X}{X}\pi-2a_1p(1-p)\phi-p(1+2a_1(1-p)\frac{F-X}{X})\Big]=0\\ (2): & -(1-p)X+\lambda\Big[2a_1p(1-p)\phi-2a_1p(1-p)\frac{Y-X}{X}\pi-(1-p)(1-2a_1p\frac{F-X}{X})\Big]=0\\ (3): & a_1p(1-p)\frac{(Y-X)^2}{X^2}\pi^2+a_1p(1-p)\phi^2-2a_1p(1-p)\frac{Y-X}{X}\pi\phi-p\frac{Y-X}{X}(1+2a_1(1-p)\frac{F-X}{X})\pi+\\ & \quad -(1-p)(1-2a_1p\frac{F-X}{X})\phi+(1-p)\frac{F-X}{X}+a_0+a_1p(1-p)\frac{(F-X)^2}{X^2}=0 \end{cases}$$

We add (1') and (2) and we get

$$\begin{cases} (1'): & -pX+\lambda\Big[2a_1p(1-p)\frac{Y-X}{X}\pi-2a_1p(1-p)\phi-p(1+2a_1(1-p)\frac{F-X}{X})\Big]=0\\ (2'): & -X+\lambda\Big[-p(1-2a_1(1-p)\frac{F-X}{X})-(1-p)(1+2a_1p\frac{F-X}{X})\Big]=0\\ (3): & a_1p(1-p)\frac{(Y-X)^2}{X^2}\pi^2+a_1p(1-p)\phi^2-2a_1p(1-p)\frac{Y-X}{X}\pi\phi-p\frac{Y-X}{X}(1+2a_1(1-p)\frac{F-X}{X})\pi+\\ & \quad -(1-p)(1-2a_1p\frac{F-X}{X})\phi+(1-p)\frac{F-X}{X}+a_0+a_1p(1-p)\frac{(F-X)^2}{X^2}=0 \end{cases}$$

$$\begin{cases} (1'): & -pX+\lambda\Big[2a_1p(1-p)\frac{Y-X}{X}\pi-2a_1p(1-p)\phi-p(1+2a_1(1-p)\frac{F-X}{X})\Big]=0\\ & \lambda=-X\\ (3): & a_1p(1-p)\frac{(Y-X)^2}{X^2}\pi^2+a_1p(1-p)\phi^2-2a_1p(1-p)\frac{Y-X}{X}\pi\phi-p\frac{Y-X}{X}(1+2a_1(1-p)\frac{F-X}{X})\pi+\\ & \quad -(1-p)(1-2a_1p\frac{F-X}{X})\phi+(1-p)\frac{F-X}{X}+a_0+a_1p(1-p)\frac{(F-X)^2}{X^2}=0 \end{cases}$$

We replace $\lambda=-X$ into (1) and we get

$$\begin{cases} & \phi=\frac{Y-X}{X}\pi-\frac{F-X}{X}\\ & \lambda=-X\\ (3): & a_1p(1-p)\frac{(Y-X)^2}{X^2}\pi^2+a_1p(1-p)\phi^2-2a_1p(1-p)\frac{Y-X}{X}\pi\phi-p\frac{Y-X}{X}(1+2a_1(1-p)\frac{F-X}{X})\pi+\\ & \quad -(1-p)(1-2a_1p\frac{F-X}{X})\phi+(1-p)\frac{F-X}{X}+a_0+a_1p(1-p)\frac{(F-X)^2}{X^2}=0 \end{cases}$$

The first result we find is $\phi=\frac{Y-X}{X}\pi-\frac{F}{X^2}+\frac{1}{X}$. Since ϕ cannot exceed 1 (contraint (c5)), we also require $(Y-X)\pi-\frac{F}{X}+1\leq X$.

Then, we replace $\phi=\frac{Y-X}{X}\pi-\frac{F-X}{X}$ into (3)

$$\begin{cases} & \phi=\frac{Y-X}{X}\pi-\frac{F-X}{X}\\ & \lambda=-X\\ (3'): & a_1pq\frac{1}{X^2}(Y-X)^2\pi^2+a_1pq(\frac{Y-X}{X}\pi-\frac{F}{X^2}+\frac{1}{X})^2-2a_1pq\frac{1}{X}(Y-X)\pi(\frac{Y-X}{X}\pi-\frac{F}{X^2}+\frac{1}{X})+\\ & \quad -p\frac{Y-X}{X}(1+2a_1q(\frac{F}{X}-1))\pi-q(1-2a_1p(\frac{F}{X}-1))(\frac{Y-X}{X}\pi-\frac{F}{X^2}+\frac{1}{X})+q(\frac{F}{X}-1)\\ & \quad +a_0+a_1pq(\frac{F}{X}-1)^2=0 \end{cases}$$

Solving the solution writes

$$\begin{cases} \phi = a_0 + (1-2p)\frac{F-X}{X} \\ \pi = \frac{X}{Y-X}a_0 + 2(1-p)\frac{F-X}{Y-X} \end{cases}$$

and, because of constraints (C2) and (C3), $Y \geq 2X$ and $q + a_0 + a_1 pq \leq 1$.